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Schrödinger operators with singular interactions: a model of tunnelling resonances

P Exner[1,2] **and S Kondej**[3]

[1] Nuclear Physics Institute, Academy of Sciences, 25068 Řež near Prague, Czech Republic
[2] Doppler Institute, Czech Technical University, Břehová 7, 11519 Prague, Czech Republic
[3] Institute of Physics, University of Zielona Góra, ul. Szafrana 4a, 65246 Zielona Góra, Poland

E-mail: exner@ujf.cas.cz and skondej@if.uz.zgora.pl

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Abstract

We discuss a generalized Schrödinger operator in $L^2(\mathbb{R}^d)$, $d = 2, 3$, with an attractive singular interaction supported by a $(d-1)$-dimensional hyperplane and a finite family of points. It can be regarded as a model of a leaky quantum wire and a family of quantum dots if $d = 2$, or surface waves in the presence of a finite number of impurities if $d = 3$. We analyse the discrete spectrum, and furthermore, we show that the resonance problem in this setting can be explicitly solved; by Birman–Schwinger method it is cast into a form similar to the Friedrichs model.

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1. Introduction

The subject of this paper is a nonrelativistic quantum Hamiltonian in $L^2(\mathbb{R}^d)$, $d = 2, 3$, with a singular interaction supported by a set consisting of two parts. One is a flat manifold of dimension $d-1$, i.e. a line for $d = 2$ and a plane for $d = 3$, and the other is a finite family of points situated in general in the complement to the manifold. The corresponding generalized Schrödinger operator can be formally written as

$$-\Delta - \alpha\delta(x-\Sigma) + \sum_{i=1}^{n} \tilde{\beta}_i \delta(x - y^{(i)}), \tag{1.1}$$

where $\alpha > 0$, $\Sigma := \{(x_1, 0); x_1 \in \mathbb{R}^{d-1}\}$ and $y^{(i)} \in \mathbb{R}^d \setminus \Sigma$; the formal coupling constants of the d-dimensional δ potentials are marked by tildes because they are not the proper parameters to be used; we will discuss this point in more detail below.

The first question to be posed is about the significance of such a Hamiltonian. Operators of the type (1.1) or similar have been studied recently with the aim of describing nanostructures

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which are 'leaky' in the sense that they do not neglect quantum tunnelling, cf [7–13] and references therein, where the physical motivation is discussed in more detail. In this sense we can regard the present model with $d = 2$ as an idealized description of a quantum wire and a collection of quantum dots which are spatially separated but are close enough to each other so that electrons are able to pass through the classically forbidden zone separating them. Similarly the three-dimensional case can be given the interpretation of a description of surface states under the influence of a finite number of point perturbations.

We will first ask about the discrete spectrum of the Hamiltonian (1.1). It will be demonstrated to be always nonempty and properties of the eigenvalues in terms of the model parameters will be derived, which complements the existing knowledge about the discrete spectrum of such generalized Schrödinger operators derived in the mentioned papers and earlier, e.g., in [3].

Our main concern in this paper, however, is the scattering within our model, in particular, the question about the existence of the resonances. It is obvious that this is an important problem for generalized Schrödinger operators with the interaction supported by a noncompact manifold of a lower dimension, of which little is known at present. The simple form of the interaction support, $\Sigma \cup \Pi$ with $\Pi := \{y^{(i)}\}$, will allow us to analyse the scattering for the operator (1.1). We will achieve that by using the generalized Birman–Schwinger method which makes it possible to convert the original PDE problem into a simpler equation which in the present situation is in part integral, in part algebraic. The main insight is that the method works not only for the discrete spectrum but also it can be used to find singularities of the analytically continued resolvent. The problem can then be reduced to a finite-rank perturbation of eigenvalues embedded in the continuous spectrum, i.e. something which brings to mind the celebrated Friedrichs model, cf [14] or [6, section 3.2].

We will pay most attention to the two-dimensional case. In the next section, we will first explain how the operator (1.1) should be properly defined, then we derive a Birman–Schwinger-type expression for its resolvent. Using this information we discuss in section 3 the discrete spectrum, first for $n = 1$, then for a pair of point perturbations showing how embedded eigenvalues due to symmetry may arise and finally for a general n. In section 4, we tackle the resonance problem using the mentioned analytical continuation of the resolvent. For simplicity we consider only the cases of single perturbation, where the resonance width is found to be exponential in terms of the distance between the line and the point, and of a pair of them to illustrate how resonances can arise from symmetry breaking. We will also treat the same problem with $n = 1$ from other point of views: as a scattering of a particle transported along the line and as a decaying unstable system. Finally in section 5, we investigate the three-dimensional case. Since the analysis is similar, we restrict ourselves to describing the features which are different for $d = 3$.

2. The Hamiltonian for $d = 2$

2.1. Definition of Hamiltonian

If $d = 2$ the interaction is supported by $\Sigma \cup \Pi$ with $\Sigma := \{(x_1, 0);\, x_1 \in \mathbb{R}\}$ and $\Pi := \{y^{(i)}\}_{i=1}^{n}$, where $y^{(i)} \in \mathbb{R}^2 \setminus \Sigma$. For simplicity we also put $L^2 \equiv L^2(\mathbb{R}^2)$. The most natural way to find a self-adjoint realization of the formal expression (1.1) is to construct the Laplace operator with appropriate boundary conditions on $\Sigma \cup \Pi$. To this aim, let us consider functions $f \in W^{2,2}_{\mathrm{loc}}(\mathbb{R}^2 \setminus (\Sigma \cup \Pi)) \cap L^2$ which are continuous on Σ. For a sufficiently small positive number ρ the restriction $f \restriction_{\mathcal{C}_{\rho,i}}$ to the circle $\mathcal{C}_{\rho,i} \equiv \mathcal{C}_\rho(y_i) := \{q \in \mathbb{R}^2 : |q - y^{(i)}| = \rho\}$ is well defined. Furthermore, we will say that function f belongs to $D(\dot{H}_{\alpha,\beta})$ if and only if the function $f_{11} + f_{22}$ on $\mathbb{R}^2 \setminus (\Sigma \cup \Pi)$, where $f_{ij} := \partial^2 f/\partial x_i \partial x_j$ is an element of L^2 and the following limits:

$$\Xi_i(f) := -\lim_{\rho\to 0} \frac{1}{\ln\rho} f \restriction_{\mathcal{C}_{\rho,i}}, \qquad \Omega_i(f) := \lim_{\rho\to 0} [f \restriction_{\mathcal{C}_{\rho,i}} + \Xi_i(f)\ln\rho],$$

for $i = 1, \ldots, n$, and

$$\Xi_\Sigma(f)(x_1) := \partial_{x_2} f(x_1, 0+) - \partial_{x_2} f(x_1, 0-), \qquad \Omega_\Sigma(f)(x_1) := f(x_1, 0)$$

are finite and satisfy the relations

$$2\pi\beta_i \Xi_i(f) = \Omega_i(f), \qquad \Xi_\Sigma(f)(x_1) = -\alpha\Omega_\Sigma(f)(x_1), \tag{2.1}$$

where $\beta_i \in \mathbb{R}$. For simplicity we put $\beta \equiv (\beta_1, \ldots, \beta_n)$ in the following. Finally, we define the operator $\dot{H}_{\alpha,\beta} : D(\dot{H}_{\alpha,\beta}) \to L^2$ acting as

$$\dot{H}_{\alpha,\beta} f(x) = -\Delta f(x) \qquad \text{for} \quad x \in \mathbb{R}^2 \setminus (\Sigma \cup \Pi).$$

The integration by parts shows that $\dot{H}_{\alpha,\beta}$ is symmetric; let $H_{\alpha,\beta}$ denote its closure. To check that the latter is self-adjoint let us consider an auxiliary operator $\dot{H}_\alpha$ defined as the Laplacian with the second one of the boundary conditions (2.1) and the additional restriction $\Xi_i(f) = \Omega_i(f) = 0$ for $f \in D(\dot{H}_\alpha)$ and all $i = 1, \ldots, n$. It is straightforward to see that the operator $\dot{H}_\alpha$ is symmetric with deficiency indices (n, n), and moreover, that the first equation of (2.1) determines n symmetric linearly independent boundary conditions; thus using the standard result [5, theorem XII.30] we conclude that $H_{\alpha,\beta}$ is self-adjoint.

Remark 2.1.

(a) The parameters determining the point interactions clearly differ from the $\tilde{\beta}_i$ used in (1.1), for instance, absence of such an interaction formally means that $\beta_i = \infty$.
(b) We introduce some notation which will be required later. Let $H_\beta := H_{0,\beta}$ be defined as the Laplacian with the point interactions only. Furthermore, let $\tilde{H}_\alpha$ denote the Laplace operator with the point perturbations (supported by Π) removed; this operator formally corresponds to $H_{\alpha,\infty}$. It is well known that both these operators are self-adjoint, cf [2].

2.2. The resolvent

To perform spectral analysis of $H_{\alpha,\beta}$ we will need its resolvent. Given $z \in \rho(-\Delta) = \mathbb{C}\setminus[0, \infty)$ denote by $R(z) = (-\Delta - z)^{-1}$ the free resolvent, which is well known to be an integral operator in L^2 with the kernel

$$G_z(x, x') = \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} \frac{\mathrm{e}^{\mathrm{i}p(x-x')}}{p^2 - z}\, \mathrm{d}p = \frac{1}{2\pi} K_0(\sqrt{-z}|x - x'|), \tag{2.2}$$

where $K_0(\cdot)$ is the Macdonald function and the function $z \mapsto \sqrt{z}$ has conventionally a cut at the positive halfline. Moreover, denote by $\mathbf{R}(z)$ the integral operator with the same kernel as $R(z)$ but acting from L^2 to $W^{2,2} \equiv W^{2,2}(\mathbb{R}^2)$; as a map between these spaces it is of course unitary.

To construct the resolvent of $H_{\alpha,\beta}$ we will need two auxiliary Hilbert spaces, $\mathcal{H}_0 := L^2(\mathbb{R})$ and $\mathcal{H}_1 := \mathbb{C}^n$, and the corresponding trace maps $\tau_0 : W^{2,2} \to \mathcal{H}_0$ and $\tau_1 : W^{2,2} \to \mathcal{H}_1$ which act as

$$\tau_0 f := f \restriction_\Sigma, \qquad \tau_1 f := f \restriction_\Pi = (f \restriction_{\{y^{(1)}\}}, \ldots, f \restriction_{\{y^{(n)}\}}),$$

respectively; in analogy with the previous section the above notation indicates the appropriate restrictions. By means of τ_i we can define the canonical embeddings of $\mathbf{R}(z)$ to $\mathcal{H}_i$, i.e.

$$\mathbf{R}_{iL}(z) = \tau_i R(z) : L^2 \to \mathcal{H}_i, \qquad \mathbf{R}_{Li}(z) = [\mathbf{R}_{iL}(z)]^* : \mathcal{H}_i \to L^2, \tag{2.3}$$

and

$$\mathbf{R}_{ji}(z) = \tau_j \mathbf{R}_{Li}(z) : \mathcal{H}_i \to \mathcal{H}_j.$$

They are all expressed naturally through the free Green's function in their kernels or 'matrix elements', with the restriction of the variable range corresponding to a given $\mathcal{H}_i$.

To express the resolvent of $H_{\alpha,\beta}$ we need the operator-valued matrix

$$\Gamma(z) = [\Gamma_{ij}(z)] : \mathcal{H}_0 \oplus \mathcal{H}_1 \to \mathcal{H}_0 \oplus \mathcal{H}_1,$$

where $\Gamma_{ij}(z) : \mathcal{H}_j \to \mathcal{H}_i$ are the operators given by

$$\begin{aligned}
&\Gamma_{ij}(z)g := -\mathbf{R}_{ij}(z)g \qquad \text{for} \quad i \neq j \quad \text{and} \quad g \in \mathcal{H}_j,\\
&\Gamma_{00}(z)f := [\alpha^{-1} - \mathbf{R}_{00}(z)]f \qquad \text{if} \quad f \in \mathcal{H}_0,\\
&\Gamma_{11}(z)\varphi := \left[s_{\beta_l}(z)\delta_{kl} - G_z(y^{(k)}, y^{(l)})(1-\delta_{kl})\right]_{k,l=1}^{n}\varphi \qquad \text{for} \quad \varphi \in \mathcal{H}_1,
\end{aligned}$$

and $s_{\beta_l}(z) = \beta_l + s(z) := \beta_l + \frac{1}{2\pi}\big(\ln\frac{\sqrt{z}}{2i} - \psi(1)\big)$, where $-\psi(1) \approx 0.577$ is the Euler number, cf [2, section 1.5].

We will also need the inverse of $\Gamma(z)$. To this aim let us denote by $\mathcal{D}$ the set of $z \in \mathbb{C}$ such that $\Gamma(z)$ is boundedly invertible; as we will see $\mathcal{D}$ coincides with the resolvent set of $H_{\alpha,\beta}$. For $z \in \mathcal{D}$ the operator $\Gamma_{00}(z)$ is invertible and thus it makes sense to define $D(z) \equiv D_{11}(z) : \mathcal{H}_1 \to \mathcal{H}_1$ by

$$D(z) = \Gamma_{11}(z) - \Gamma_{10}(z)\Gamma_{00}(z)^{-1}\Gamma_{01}(z), \tag{2.4}$$

which is invertible for $z \in \mathcal{D}$; the above operator will be called the *reduced determinant* of Γ. By a straightforward calculation one can check that the inverse of $\Gamma(z)$ is given by

$$[\Gamma(z)]^{-1} : \mathcal{H}_0 \oplus \mathcal{H}_1 \to \mathcal{H}_0 \oplus \mathcal{H}_1, \tag{2.5}$$

with the 'block elements' defined by

$$\begin{aligned}
&[\Gamma(z)]_{11}^{-1} = D(z)^{-1},\\
&[\Gamma(z)]_{00}^{-1} = \Gamma_{00}(z)^{-1} + \Gamma_{00}(z)^{-1}\Gamma_{01}(z)D(z)^{-1}\Gamma_{10}(z)\Gamma_{00}(z)^{-1},\\
&[\Gamma(z)]_{01}^{-1} = -\Gamma_{00}(z)^{-1}\Gamma_{01}(z)D(z)^{-1},\\
&[\Gamma(z)]_{10}^{-1} = -D(z)^{-1}\Gamma_{10}(z)\Gamma_{00}(z)^{-1};
\end{aligned}$$

in the above formulae we use notation $\Gamma_{ij}(z)^{-1}$ for the inverse of $\Gamma_{ij}(z)$ and $[\Gamma(z)]_{ij}^{-1}$ for the matrix element of $[\Gamma(z)]^{-1}$.

With these preliminaries we are ready to state the sought formula for the explicit form of the resolvent of $H_{\alpha,\beta}$.

Theorem 2.2. *For any $z \in \rho(H_{\alpha,\beta})$ with* $\operatorname{Im} z > 0$ *we have*

$$R_{\alpha,\beta}(z) \equiv (H_{\alpha,\beta} - z)^{-1} = R(z) + \sum_{i,j=0}^{1} \mathbf{R}_{Li}(z)[\Gamma(z)]_{ij}^{-1}\mathbf{R}_{jL}(z). \tag{2.6}$$

Proof. We employ again the vector notation, $\Xi(f) \equiv (\Xi_1(f), \ldots, \Xi_n(f))$ and $\Omega(f) \equiv (\Omega_1(f), \ldots, \Omega_n(f))$. We have to check that $f \in D(H_{\alpha,\beta})$ holds if and only if $f = \tilde{R}_{\alpha,\beta}(z)g$ for some $g \in L^2$, where $\tilde{R}_{\alpha,\beta}(z)$ denotes the operator on the right-hand side of the last equation. Suppose that f is of this form. It belongs obviously to $W^{2,2}_{\mathrm{loc}}(\mathbb{R}^2 \setminus (\Sigma \cup \Pi)) \cap L^2$ because the same is true for all its components. Combining the definitions of $\mathbf{R}_{ij}$, $[\Gamma(z)]_{ij}^{-1}$, and the functionals Ξ_i and Ω_i introduced above with the asymptotic behaviour of the Macdonald function, specifically

$$K_0(\sqrt{-z}\rho) \to -\ln\rho - 2\pi s(z) + \mathcal{O}(\rho) \qquad \text{for} \quad \rho \to 0, \tag{2.7}$$

we arrive at

$$2\pi\,\Xi(f) = \sum_{i=0}^{1}[\Gamma(z)]_{1i}^{-1}\mathbf{R}_{iL}(z)g,$$

$$\Omega(f) = \mathbf{R}_{1L}(z)g - \sum_{i=0}^{1}\Gamma_{10}(z)[\Gamma(z)]_{0i}^{-1}\mathbf{R}_{iL}g - s(z)\sum_{i=0}^{1}[\Gamma(z)]_{1i}^{-1}\mathbf{R}_{iL}(z)g.$$

Let us consider separately the components of $\Xi(f), \Omega(f)$ coming from the behaviour of g at the points of the set Π and on Σ, i.e. for $i = 1, 2$, which means to define the vectors $\Xi^i(f) := \frac{1}{2\pi}[\Gamma(z)]_{1i}^{-1}\mathbf{R}_{iL}g$ and

$$\Omega^0(f) := \big[-\Gamma_{10}(z)[\Gamma(z)]_{00}^{-1} - s(z)[\Gamma(z)]_{10}^{-1}\big]\mathbf{R}_{0L}g,$$

$$\Omega^1(f) := \big[1 - \Gamma_{10}(z)[\Gamma(z)]_{01}^{-1} - s(z)[\Gamma(z)]_{11}^{-1}\big]\mathbf{R}_{1L}g.$$

Using the properties of $[\Gamma_{ij}(z)]$ and its inverse it is straightforward to check that $\Omega_k^i(f) = 2\pi\beta_k\Xi_k^i(f)$ holds for $i = 0, 1$ and $k = 1, \ldots, n$; the symbols $\Omega_k^i(f)$, $\Xi_k^i(f)$ mean here the kth component of $\Omega^i(f)$, $\Xi^i(f)$ respectively. Similar calculations yield the relation $\Xi_\Sigma(f) = -\alpha\Omega_\Sigma(f)$ which shows that f belongs to $D(H_{\alpha,\beta})$, and the converse statement, namely that any function from $D(H_{\alpha,\beta})$ admits a representation of the form $f = \tilde{R}_{\alpha,\beta}(z)g$. To conclude the proof, observe that for a function $f \in D(H_{\alpha,\beta})$ which vanishes on $\Sigma \cup \Pi$ we have $(-\Delta - z)f = g$. Consequently, $\tilde{R}_{\alpha,\beta}(z) = R_{\alpha,\beta}(z)$ is the resolvent of the Laplace operator in L^2 with the boundary conditions (2.1). □

2.3. Another form of the resolvent

With a later purpose in mind it is useful to look at the model in question also from another point of view, namely as a point-interaction perturbation of the 'line only' Hamiltonian $\tilde{H}_\alpha$. In the same way as above we can check that the resolvent of $\tilde{H}_\alpha$ is the integral operator

$$R_\alpha(z) = R(z) + R_{L0}(z)\Gamma_{00}^{-1}R_{0L}(z), \tag{2.8}$$

for any given $z \in \rho(\tilde{H}_\alpha) = \mathbb{C}\backslash\big[-\frac{1}{4}\alpha^2, \infty\big)$. Define now the operators $\mathbf{R}_{\alpha;L1}(z) : \mathcal{H}_1 \to L^2$ and $\mathbf{R}_{\alpha;1L}(z) : L^2 \to \mathcal{H}_1$ by

$$\mathbf{R}_{\alpha;1L}(z)f = R_\alpha(z)f \restriction_\Pi \qquad \text{for} \quad f \in L^2 \tag{2.9}$$

and $\mathbf{R}_{\alpha;L1}(z) = \mathbf{R}_{\alpha;1L}^*(z)$. The Hamiltonian $H_{\alpha,\beta}$ is obtained by adding a finite number of point perturbations to $\tilde{H}_\alpha$. Consequently, the difference of the resolvents $R_{\alpha,\beta}$ and R_α is given by Krein's formula

$$R_{\alpha,\beta}(z) = R_\alpha(z) + \mathbf{R}_{\alpha;L1}(z)\Gamma_{\alpha;11}(z)^{-1}\mathbf{R}_{\alpha;1L}(z),$$

with

$$\Gamma_{\alpha;11}(z)\varphi = \big(s_{\beta,k}^{(\alpha)}(z)\delta_{kl} - G_z^{(\alpha)}(y^{(k)}, y^{(l)})(1-\delta_{kl})\big)\varphi \qquad \text{for} \quad \varphi \in \mathcal{H}_1,$$

where $s_{\beta,k}^{(\alpha)}(z) := \beta_k - \lim_{\eta\to 0}\big(G_z^{(\alpha)}(y^{(k)}, y^{(k)}+\eta) + \frac{1}{2\pi}\ln|\eta|\big)$ and $G_z^{(\alpha)}$ is the integral kernel of the operator $R_\alpha(z)$. In fact, this can be simplified as follows:

Proposition 2.3. *For any* $z \in \rho(H_{\alpha,\beta})$ *with* $\operatorname{Im} z > 0$ *we have*

$$R_{\alpha,\beta}(z) = R_\alpha(z) + \mathbf{R}_{\alpha;L1}(z)D(z)^{-1}\mathbf{R}_{\alpha;1L}(z).$$

Proof. Using the asymptotic behaviour of the Macdonald function we get

$$s_{\beta,k}^{(\alpha)}(z) = s_{\beta_k}(z) - (\mathbf{R}_{10}(z)\Gamma_{00}(z)^{-1}\mathbf{R}_{01}(z))_{kk}.$$

This yields $\Gamma_{\alpha;11}(z) = D(z)$, and thus the claim of the proposition. □

3. Spectral analysis

We begin the spectral analysis of $H_{\alpha,\beta}$ by localizing the essential spectrum. To this aim let us consider the auxiliary 'line-only' operator $\tilde{H}_\alpha$ introduced above. Separating variables and using the fact that a one-dimensional Laplace operator with a single point interaction of coupling constant α has just one isolated eigenvalue (recall that $\alpha > 0$) equal to $-\frac{1}{4}\alpha^2$ we find that $\sigma(\tilde{H}_\alpha) = \sigma_{\mathrm{ac}}(\tilde{H}_\alpha) = \left[-\frac{1}{4}\alpha^2, \infty\right)$. The point interactions in $H_{\alpha,\beta}$ represent by proposition 2.3 a finite-rank perturbation of the resolvent, hence the essential spectrum is preserved by Weyl's theorem. Moreover, the explicit expression of the resolvent makes it possible to employ [19, theorem XIII.19] to conclude that the singularly continuous spectrum of $H_{\alpha,\beta}$ is empty, i.e. that

$$\sigma_{\mathrm{ess}}(H_{\alpha,\beta}) = \sigma_{\mathrm{ac}}(H_{\alpha,\beta}) = \left[-\tfrac{1}{4}\alpha^2, \infty\right). \tag{3.1}$$

To demonstrate the existence of isolated points of the spectrum for $H_{\alpha,\beta}$ and to find the corresponding eigenvectors we employ the following equivalences:

$$z \in \sigma_{\mathrm{d}}(H_{\alpha,\beta}) \quad \Leftrightarrow \quad 0 \in \sigma_{\mathrm{d}}(\Gamma(z)), \qquad \dim\ker\Gamma(z) = \dim\ker(H_{\alpha,\beta} - z), \tag{3.2}$$

$$H_{\alpha,\beta}\phi_z = z\phi_z \quad \Leftrightarrow \quad \phi_z = \sum_{i=0}^{1} \mathbf{R}_{Li}(z)\eta_{i,z} \qquad \text{for} \quad z \in \sigma_{\mathrm{disc}}(H_{\alpha,\beta}), \tag{3.3}$$

where $(\eta_{0,z}, \eta_{1,z}) \in \ker\Gamma(z)$. They are nothing else than a generalization of the Birman–Schwinger principle to the situation when the interaction in the Schrödinger operator in question is singular and supported by a zero-measure set; in the present form they follow from an abstract result of [18, theorem 3.4]. Thus, to investigate the discrete spectrum it suffices to study zeros of the operator-valued function $z \mapsto \Gamma(z)$. This will be the starting point for considerations in the rest of this section.

3.1. Discrete spectrum for one point interaction

We start with the simplest case when the interaction in $H_{\alpha,\beta}$ is supported by Σ and at a single point y. In such a case, of course, we can choose $y = (0, a)$ with $a > 0$ without loss of generality. As indicated above the spectrum in $\left[-\frac{1}{4}\alpha^2, \infty\right)$ is purely absolutely continuous; our aim is to show that $H_{\alpha,\beta}$ has always exactly one isolated eigenvalue and to investigate its dependence on the distance a between y and Σ. In particular, we will show that the eigenvalue behaviour for large a basically depends on whether the number

$$\epsilon_\beta = -4\,\mathrm{e}^{2(-2\pi\beta+\psi(1))}, \tag{3.4}$$

where $-\psi(1) \approx 0.577$ is the Euler number, belongs to the absolutely continuous spectrum or not; recall that ϵ_β is the only isolated eigenvalue of the point-interaction Hamiltonian H_β, cf [2, section 1.5].

Since zeros of $\Gamma(z)$ determine eigenvalues of $H_{\alpha,\beta}$, it is convenient to rewrite the operator $\Gamma(z)$ in a more explicit form. It is straightforward to see that its part $\Gamma_{00}(z)$ acts in the momentum representation as a simple multiplication, and therefore

$$\Gamma_{00}(z)f(x) = \frac{1}{(2\pi)^{1/2}} \int_{\mathbb{R}} \left[\frac{1}{\alpha} - \frac{\mathrm{i}}{2(z-p^2)^{1/2}}\right] \tilde{f}(p)\,\mathrm{e}^{\mathrm{i}px}\,\mathrm{d}p.$$

Moreover, using the expression for the Green function of the one-dimensional Laplace operator,

$$\frac{1}{2\pi}\int_{\mathbb{R}} \frac{\mathrm{e}^{\mathrm{i}px}}{p^2 - z}\,\mathrm{d}p = \frac{\mathrm{i}}{2\sqrt{z}}\,\mathrm{e}^{\mathrm{i}\sqrt{z}|x|}, \tag{3.5}$$

we can express the 'off-diagonal' operator components as

$$(\Gamma_{01}(z)\phi)(x) = \nu_z^+(x)\phi, \qquad \Gamma_{10}(z)f = \int_{\mathbb{R}} \nu_z^-(x) f(x)\,\mathrm{d}x, \tag{3.6}$$

for $\phi \in \mathcal{H}_1$ and $f \in \mathcal{H}_0$, respectively, where

$$\nu_z^{\pm}(x) := \int_{\mathbb{R}} \upsilon_z(p)\,\mathrm{e}^{\pm\mathrm{i}px}\,\mathrm{d}p, \qquad \upsilon_z(p) := \frac{\mathrm{i}}{4\pi}\frac{\mathrm{e}^{\mathrm{i}(z-p^2)^{1/2}a}}{(z-p^2)^{1/2}}. \tag{3.7}$$

While later we will consider analytic continuation of some of the resolvent 'constituents', with operators (3.6) it is sufficient to stay at the first sheet of $z \mapsto (z-p^2)^{1/2}$, i.e. to suppose that $\mathrm{Im}(z-p^2)^{1/2} > 0$. In that case the functions $\nu_z^{\pm}$ belong to $\mathcal{H}_0$, and consequently, the 'off-diagonal' operators, $\Gamma_{ij}(z)$ with $i \neq j$, are well defined.

To proceed further we make two observations. The first is the equivalence

$$0 \in \sigma_{\mathrm{d}}(\Gamma(z)) \Leftrightarrow 0 \in \sigma_{\mathrm{d}}(D(z)),$$

where $D(z)$ is the reduced determinant of $\Gamma(z)$ given by (2.4); this means that it suffices to investigate zeros of the map $z \mapsto D(z)$. Secondly, as we know that $H_{\alpha,\beta}$ is self-adjoint, we can restrict ourselves to $z = -\kappa^2$ with $\kappa > 0$. For convenience we introduce the abbreviations $\check{\Gamma}(\kappa) := \Gamma(-\kappa^2)$, $\check{D}(\kappa) = D(-\kappa^2)$, and the analogous symbols for other functions. By a straightforward computation using formulae (3.7), (3.6) one can check that $\check{D}(\kappa)$ is an operator of multiplication, $\check{D}(\kappa)\varphi = \check{d}(\kappa)\varphi$, by the number

$$\check{d}(\kappa) \equiv \check{d}_a(\kappa) := \check{s}_\beta(\kappa) - \check{\phi}_a(\kappa),$$

where

$$\check{\phi}_a(\kappa) := \frac{\alpha}{4\pi}\int_{\mathbb{R}} \frac{\mathrm{e}^{-2(p^2+\kappa^2)^{1/2}a}}{(2(p^2+\kappa^2)^{1/2}-\alpha)(p^2+\kappa^2)^{1/2}}\,\mathrm{d}p \tag{3.8}$$

and

$$\check{s}_\beta(\kappa) = s_\beta(-\kappa^2) := \beta + \frac{1}{2\pi}\left[\ln\frac{\kappa}{2} - \psi(1)\right].$$

Consequently, roots of the equation

$$\check{d}_a(\kappa) = 0 \qquad \text{for} \quad \kappa \in (\alpha/2, \infty) \tag{3.9}$$

determine through $z = -\kappa^2$ the discrete spectrum of $H_{\alpha,\beta}$.

Now we are ready to state a claim which characterizes the discrete spectrum of $H_{\alpha,\beta}$ in the case of a single point perturbation.

Theorem 3.1. *For given $\alpha > 0$ and $\beta \in \mathbb{R}$ the operator $H_{\alpha,\beta}$ has exactly one isolated eigenvalue $-\kappa_a^2$ with the eigenvector which can be represented by*

$$\mathrm{const}\int_{\mathbb{R}^2}\left(\frac{\mathrm{e}^{-\mathrm{i}p_2a}}{2\pi} + \frac{\alpha\,\mathrm{e}^{-(p_1^2+\kappa_a^2)^{1/2}a}}{2\left(p_1^2+\kappa_a^2\right)^{1/2}-\alpha}\right)\frac{\mathrm{e}^{\mathrm{i}px}}{p^2+\kappa_a^2}\,\mathrm{d}p, \tag{3.10}$$

where we integrate with respect to $p = (p_1, p_2)$.

Proof. To check that there is a κ_a satisfying (3.9), it suffices to investigate the behaviour of $\check{d}_a$ at infinity and near the number $\frac{1}{2}\alpha$. Using the above definitions of $\check{s}_\beta$ and $\check{\phi}_a$ it is easy to see that the function $\kappa \mapsto \check{d}_a(\kappa)$ is strictly increasing with the limits $\check{d}_a(\kappa) \to \pm\infty$ as $\kappa \to \infty$ and $\kappa \to \frac{1}{2}\alpha+$, respectively. Thus, there is exactly one $\kappa_a \in \left(\frac{1}{2}\alpha, \infty\right)$ such that $\check{d}_a(\kappa_a) = 0$. Formula (3.10) can be obtained directly from (3.3). $\square$

Next we want to look at the asymptotic behaviour of the eigenvalue the existence of which we have just established for large as well as small distance a; in this respect it is convenient to use the notation $H_{\alpha,\beta,a}$ for the operator in question. The answer is again contained in the behaviour of the functions $\check{s}_\beta(\cdot)$, and $\check{\phi}_a(\cdot)$. Given $\kappa \in \left(\frac{1}{2}\alpha, \infty\right)$ we define the function $a \mapsto \tilde{\phi}_\kappa(a) = \check{\phi}_a(\kappa)$; using (3.8) it is easy to see that it is decreasing on the indicated interval. Combining this with the fact that $\check{s}_\beta(\cdot)$ is increasing we come to the conclusion that the function $a \mapsto \kappa_a$ is decreasing on $(0, \infty)$. To determine its behaviour at the endpoints of the interval let us note that

$$\lim_{a\to\infty} \tilde{\phi}_\kappa(a) = 0.$$

This limit in combination with the relation $\check{s}_\beta(\sqrt{-\epsilon_\beta}) = 0$, where ϵ_β is the point-interaction eigenvalue given by (3.4), yields

$$\lim_{a\to\infty} \kappa_a = \sqrt{-\epsilon_\beta} \qquad \text{if} \quad \sqrt{-\epsilon_\beta} \in (\alpha/2, \infty)$$

and

$$\lim_{a\to\infty} \kappa_a = \frac{\alpha}{2} \qquad \text{if} \quad \sqrt{-\epsilon_\beta} \in (-\infty, \alpha/2].$$

Let us turn next to the behaviour $a \mapsto \kappa_a$ for small a. To this aim we note that for a fixed κ the expression

$$\check{\phi}_0(\kappa) := \frac{\alpha}{4\pi} \int_{\mathbb{R}} \frac{1}{(2(p^2+\kappa^2)^{1/2} - \alpha)(p^2+\kappa^2)^{1/2}} \, \mathrm{d}p$$

provides an upper bound for $\check{\phi}_a(\kappa)$. It is straightforward to check that $\check{\phi}_0(\kappa) \to 0$ as $\kappa \to \infty$ and $\check{\phi}_0(\kappa) \to \infty$ as $\kappa \to \frac{1}{2}\alpha+$. It follows that there is a number $\kappa_0 \in \left(\frac{1}{2}\alpha, \infty\right)$ which is a solution of $\check{s}_\beta(\kappa) - \check{\phi}_0(\kappa) = 0$ and provides an upper bound to the function $a \mapsto \kappa_a$. These considerations can be summarized as follows:

Theorem 3.2. *The eigenvalue $-\kappa_a^2$ of $H_{\alpha,\beta,a}$ is increasing as a function of the distance a. Moreover, we have*

$$-\lim_{a\to\infty} \kappa_a^2 = \epsilon_\beta \qquad \text{if} \quad \epsilon_\beta \in \left(-\infty, -\tfrac{1}{4}\alpha^2\right]$$

and

$$-\lim_{a\to\infty} \kappa_a^2 = -\tfrac{1}{4}\alpha^2 \qquad \text{if} \quad \epsilon_\beta \in \left(-\tfrac{1}{4}\alpha^2, \infty\right).$$

On the other hand, $-\kappa_0^2$ is the best lower bound for $-\kappa_a^2$, i.e. we have

$$-\lim_{a\to 0} \kappa_a^2 = -\kappa_0^2.$$

3.2. A mirror-symmetric pair of point interactions

Generally speaking, the case of $n = 2$ can be treated within the discussion of the discrete spectrum of $H_{\alpha,\beta}$ with $n > 1$ presented in the next subsection. Here we single out the situation where the system has a mirror symmetry to illustrate that it can give rise to eigenvalues embedded in the continuous spectrum. To be specific, we assume that the interaction sites are located symmetrically with respect to the line Σ, i.e. $x_1 = (0, a)$, $x_2 = (0, -a)$ with some $a > 0$, and moreover, the coupling strengths are the same, $\beta_1 = \beta_2 = \beta$.

As in the case $n = 1$, the relation between the number $-\frac{1}{4}\alpha^2$ and the point-interaction eigenvalues will be important for spectral properties. Consider the system with the line component of the interaction removed which is described by the operator H_β. It has

$\sigma_{ac}(H_\beta) = [0, \infty)$ and at least one and at most two eigenvalues. Let us denote them μ_1, μ_2 and assume that $\mu_1 < \mu_2$; if there exists only one eigenvalue we put $\mu_2 := 0$. From the explicit resolvent formula [2, section II.4] it follows that $\mu_i = -\kappa_i^2$, where κ_i are solutions of the equation

$$\check{s}_\beta(\kappa)^2 - K_0(2\kappa a)^2 = 0, \qquad \kappa > 0,$$

which implies the inequalities

$$\mu_1 < \epsilon_\beta < \mu_2; \tag{3.11}$$

they follow also from Dirichlet–Neumann bracketing [19, section XIII.15] and it is useful to note that the number μ_1, μ_2 is the eigenvalue corresponding to the symmetric and antisymmetric eigenfunctions of H_β, respectively.

To find the isolated eigenvalue of $H_{\alpha,\beta}$, we will employ the BS-principle expressed by (3.2). Proceeding similarly as in the previous section we show that the number $-\tilde{\kappa}^2$ is an eigenvalue of $H_{\alpha,\beta}$ *iff* $\tilde{\kappa}$ is a solution of

$$\check{d}(\kappa) = 0 \qquad \text{for} \quad \kappa \in (\alpha/2, \infty), \tag{3.12}$$

where the function $\check{d}(\cdot)$ means the determinant of $\check{D}(\cdot)$ being thus given by

$$\check{d}(\kappa) = (\check{s}_\beta(\kappa) + K_0(2\kappa a))(\check{s}_\beta(\kappa) - K_0(2\kappa a) - 2\check{\phi}_a(\kappa))$$

and $\check{d}(\kappa)$ is again given by (3.8), i.e.

$$\check{\phi}_a(\kappa) = \frac{\alpha}{4\pi} \int_{\mathbb{R}} \frac{\mathrm{e}^{-2(p^2+\kappa^2)^{1/2}a}}{(2(p^2+\kappa^2)^{1/2} - \alpha)(p^2+\kappa^2)^{1/2}}\, \mathrm{d}p. \tag{3.13}$$

Now we can describe the point spectrum of $H_{\alpha,\beta}$ in the given situation.

Theorem 3.3. *$H_{\alpha,\beta}$ has always at least one isolated eigenvalue. Moreover,*

(i) if $-\frac{1}{4}\alpha^2 < \mu_2 < 0$, then $H_{\alpha,\beta}$ has one isolated eigenvalue and one embedded eigenvalue which is equal to μ_2,

(ii) on the other hand, if $\mu_2 < -\frac{1}{4}\alpha^2$, then $H_{\alpha,\beta}$ has two isolated eigenvalues the larger of which is given by μ_2.

Proof. Using the behaviour of functions $\check{s}_\beta$, K_0 and $\check{\phi}_a$ at infinity and near the number $\frac{1}{2}\alpha$ we can conclude that the equation

$$\check{s}_\beta(\kappa) - K_0(2\kappa a) - 2\check{\phi}_a(\kappa) = 0$$

coming from the second factor in the spectral condition has for any parameter values exactly one solution in $\left(\frac{1}{2}\alpha, \infty\right)$ which naturally solves also (3.12); this means that the operator $H_{\alpha,\beta}$ has always at least one isolated eigenvalue. Moreover, if $\mu_2 < -\frac{1}{4}\alpha^2$ equation (3.12) has one more solution given by the number κ_2; this completes the proof of (ii). Assume next $-\frac{1}{4}\alpha^2 < \mu_2 < 0$. As we have already mentioned the number μ_2 is the eigenvalue of H_β corresponding to eigenfunction ψ_{μ_2} antisymmetric w.r.t. Σ. It is easy to see that $\psi_{\mu_2} \in D(H_{\alpha,\beta})$ and both the boundary functions $\Xi_\Sigma(\psi_{\mu_2}), \Omega_\Sigma(\psi_{\mu_2})$ vanish. This implies $H_{\alpha,\beta}\psi_{\mu_2} = H_\beta\psi_{\mu_2}$, in other words that ψ_{μ_2} is at the same time an eigenvector of $H_{\alpha,\beta}$ corresponding to μ_2. □

Remark 3.4. Let us note here that the condition

$$\epsilon_\beta > -\tfrac{1}{4}\alpha^2 \tag{3.14}$$

is sufficient for $\mu_2 > -\frac{1}{4}\alpha^2$ in view of (3.11), while the converse statement is not true in general. It may happen when the distance a is sufficiently small that even if ϵ_β is below the threshold of the essential spectrum, the number μ_2 would satisfy $\mu_2 > -\frac{1}{4}\alpha^2$ so according to theorem 3.3 it will appear in the spectrum of $H_{\alpha,\beta}$ as an embedded eigenvalue.

3.3. Finitely many point interactions

Let us finally turn to analysis of the discrete spectrum in the general case with finitely many points of interaction and coupling constants determined by components of the vector $\beta = (\beta_1, \ldots, \beta_n)$. We assume that the perturbations are located at $y^{(i)} = (l_i, a_i)$, where $l_i \in \mathbb{R}$, $a_i \in \mathbb{R} \setminus \{0\}$, and denote by

$$d_{ij} := |y^{(i)} - y^{(j)}|$$

the distances between them. Our strategy will be similar to before, namely, to recover the discrete spectrum of $H_{\alpha,\beta}$ we will employ the equivalence (3.2) which allows us to describe eigenvalues of $H_{\alpha,\beta}$ in terms of the zeros of $z \mapsto \Gamma(z)$. This in turn can be reduced to the problem of finding zeros of the $n \times n$ matrix $D(z) := \Gamma_{11}(z) - \Gamma_{10}(z)\Gamma_{00}(z)^{-1}\Gamma_{01}(z) : \mathcal{H}_1 \to \mathcal{H}_1$ for z negative. To proceed further we introduce the notation $\Gamma_{j;0}$ for the jth component of Γ_{10} and $\Gamma_{i;j}$ for the corresponding matrix element of Γ_{11}. We also introduce the following auxiliary functions of $z \in \mathbb{C} \setminus \left[-\frac{1}{4}\alpha^2, \infty\right)$:

$$\Theta^j_{i_1} := \Gamma_{j;0}\Gamma_{00}^{-1}\Gamma_{0;i_1},$$

$$A^j_{i_2,\ldots,i_k} := \begin{cases} \Gamma_{1;i_2} \cdots \Gamma_{j-1;i_j}\Gamma_{j+1;i_{j+1}} \cdots \Gamma_{k;i_k} & \text{if} \quad j > 1, \\ \Gamma_{2;i_2} \cdots \Gamma_{k;i_k} & \text{if} \quad j = 1. \end{cases}$$

A straightforward computation shows that the determinant of $D(\cdot)$ is given by the function $d(\cdot)$ with the values

$$d(z) = \sum_{\pi \in \mathcal{P}_n} \operatorname{sgn} \pi \left(\sum_{j=1}^{n} (-1)^j S^j_{p_1,\ldots,p_n} + \Gamma_{1;p_1} \cdots \Gamma_{n;p_n} \right)(z), \tag{3.15}$$

where $S^j_{p_1,\ldots,p_n} := \Theta^j_{p_1} A^j_{p_2,\ldots,p_n}$, $\mathcal{P}_n$ is the permutation group of $(1, \ldots, n)$ and $\pi = (p_1, \ldots, p_n)$ is an element of $\mathcal{P}_n$. Since we are interested in the negative part of the spectrum we put $\check{d}(\kappa) = d(-\kappa^2)$ and the same convention will be kept for the other expressions. According to the above general discussion, the eigenvalues of $H_{\alpha,\beta}$ are determined by solution of the equation

$$\check{d}(\kappa) = 0 \qquad \text{for} \quad \kappa \in (\alpha/2, \infty). \tag{3.16}$$

To concretize the function $\check{d}(\cdot)$ we need more information about the functions involved in the definition of $D(\cdot)$. We have

$$\check{\Theta}^j_k(\kappa) = \frac{\alpha}{4\pi} \int_{\mathbb{R}} \frac{\mathrm{e}^{-(p^2+\kappa^2)^{1/2}(|a_i|+|a_j|)}}{(2(p^2+\kappa^2)^{1/2} - \alpha)(p^2+\kappa^2)^{1/2}} \mathrm{e}^{\mathrm{i}p(l_j-l_k)}\, \mathrm{d}p \tag{3.17}$$

and

$$\check{\Gamma}_{j;k}(\kappa) = -\frac{1}{2\pi} K_0(d_{jk}\kappa) \qquad \text{for} \quad j \neq k; \tag{3.18}$$

recall that the diagonal elements for $j \geqslant 1$ are given by the numbers $\check{\Gamma}_{j;j}(\kappa) = \check{s}_{\beta_j}(\kappa)$. After these preliminaries we are ready to prove the following theorem:

Theorem 3.5. *Let $\beta = (\beta_1, \ldots, \beta_n)$, where $\beta_i \in \mathbb{R}$ and $\alpha > 0$. The operator $H_{\alpha,\beta}$ has at least one isolated eigenvalue and at most n of them; they are determined by solutions of equation (3.16). In particular, if all the numbers $-\beta_i$ are sufficiently large then $H_{\alpha,\beta}$ has exactly n eigenvalues.*

Proof. Let us consider again the operator $\dot{H}_\alpha$ defined in section 2.1. Since it is symmetric with deficiency indices (n, n) and $\dot{H}_\alpha \geqslant -\frac{1}{4}\alpha^2$, there are at most n eigenvalues of $H_{\alpha,\beta}$, cf [20, section 8.4]. The remaining part of the proof will be divided into four steps.

Step 1. We will show that if all the numbers β_i are sufficiently large then equation (3.16) has at least one solution. To this aim we shall investigate the behaviour of $\check{d}(\cdot)$ at infinity and near the number $\frac{1}{2}\alpha$. It is easy to see that for large values of the argument κ, the behaviour of the function $\check{d}$ is determined by the term $\prod_{i=1}^{n} \check{\Gamma}_{i;i} = \prod_{i=1}^{n} \check{s}_{\beta_i}$; this implies

$$\check{d}(\kappa) \to \infty \qquad \text{as} \quad \kappa \to \infty. \tag{3.19}$$

On the other hand, the function $\check{d}$ has a singularity at $\frac{1}{2}\alpha$ induced by $\check{\Theta}^i_j$. This fact allows us to conclude that if all the numbers β_i are sufficiently large then the behaviour of $\check{d}(\kappa)$ near $\frac{1}{2}\alpha$ is dominated by the components of $-S^j_{j,1,2,\ldots,j-1,j+1,\ldots,n}$ which look like $-\check{\Theta}^j_j \beta_1 \cdots \beta_{j-1}\beta_{j+1} \cdots \beta_n$. Since they are all negative under our assumption, we arrive at

$$\check{d}(\kappa) \to -\infty \qquad \text{as} \quad \kappa \to \tfrac{1}{2}\alpha.$$

Combining this with (3.19) we demonstrate the existence of at least one solution of (3.16) if the coupling constants β_i are sufficiently large.

Step 2. Note further that the functions $\check{\Gamma}_{i;i} = \check{s}_{\beta_i}$ are increasing with respect to each parameter β_i while the other matrix elements of $\check{\Gamma}$ are independent of all the β_i. Combining this with the minimax principle and the results obtained in the previous step we find that for all $\beta_1, \ldots, \beta_n$ there exists at least one solution of (3.16), and consequently, an eigenvalue of $H_{\alpha,\beta}$.

Step 3. Let $\tilde{\kappa}$ be a solution to (3.16). From (3.17) and (3.18) in combination with the explicit expression for $\check{s}_{\beta_i}$ one finds that if all the coupling constants $\beta_i \to -\infty$ then $\tilde{\kappa}$ tends to infinity or to the number $\frac{1}{2}\alpha$. However, the latter is excluded by the monotonicity proved in the previous step. Thus we obtain

$$\tilde{\kappa} \to \infty \qquad \text{as} \quad \beta_i \to -\infty \qquad \text{for} \quad i = 1, \ldots, n.$$

Step 4. Using the explicit formulae for operators $\check{\Gamma}_{i,j}$ one checks that operator $\check{\Gamma}(\kappa)$ approaches $\check{S}(\kappa)$ in the norm operator sense as $\kappa \to \infty$, where $\check{S}(\kappa)$ is the operator-valued diagonal matrix given by

$$\begin{aligned} &\check{S}_{00}(\kappa) = 0, \\ &\check{S}_{11}(\kappa) = \left[\check{s}_{\beta_k}(\kappa)\delta_{kl}\right]_{k,l=1}^{n}, \\ &\check{S}_{ij}(\kappa) = 0 \qquad \text{for} \quad i, j = 0, 1 \quad \text{and} \quad i \neq j. \end{aligned}$$

Since there exist n solutions of the operator equation $\check{S}(\kappa) = 0$ we arrive at the final conclusion that for $-\beta_i$ sufficiently large the operator $H_{\alpha,\beta}$ has the 'full number' n of isolated eigenvalues. □

4. Resonances

Determining the spectrum as a set does not exhaust the interesting properties of the present model; we now turn to features 'hidden' in the continuous component (3.1). We will concentrate on the negative part of this interval, where in the absence of point perturbations we have a simple one-dimensional transport: the wavefunctions factorize into the transverse factor which is the eigenfunction of the one-dimensional point interaction, and the longitudinal one which is a wave packet moving and spreading in the usual way. If we now add point perturbation(s) the transport may be affected by tunnelling between the line and these singular 'potential wells', at least if such a process is energetically allowed; our goal stated in the introduction is to show the existence of 'tunnelling' resonances and to find their properties. For the sake of simplicity we shall consider mostly (with the exception of section 4.2) the case when the Hamiltonian $H_{\alpha,\beta}$ has a single point perturbation.

Following the standard ideology, to find resonances we have to construct the analytical continuation of $z \mapsto R_{\alpha,\beta}(z)$ to the second sheet across the cut corresponding to the continuous spectrum and to find poles of this continuation. Our main insight is that the constituents of the operator on the right-hand side of (2.6) can be separately continued analytically, and moreover as we remarked above, for the factors (3.6) in fact no continuation is needed, i.e. we may suppose that $\mathrm{Im}(z-p^2)^{1/2} > 0$. Thus we have to deal only with the middle factor in the interaction term of (2.6), in other words, we can extend the Birman–Schwinger principle to the complex region and to look for zeros in the analytic continuation of $D(\cdot)$. Taking into account the structure of the auxiliary space $\mathcal{H}_0 \oplus \mathcal{H}_1$ we get in this way a problem reminiscent of the Friedrichs model, cf [14], or [6, section 3.2] for a review.

4.1. Resonance for $H_{\alpha,\beta}$ with a single point interaction

The Friedrichs model analogy suggests treating our problem perturbatively assuming that in the 'decoupled' case which corresponds here to the limit $a \to \infty$ we have the point-interaction eigenvalue ϵ_β embedded in the continuous spectrum. Following the above sketched programme we first note that by formulae (3.5), (3.6) and (3.7) the operator-valued function $z \mapsto D(z)$, $z \in \mathbb{C}\setminus\left[-\frac{1}{4}\alpha, \infty\right)$ is now one dimensional, i.e. a multiplication by the function

$$d_a(z) = s_\beta(z) - \phi_a(z), \qquad \text{where} \quad \phi_a(z) := \int_0^\infty \frac{\mu(z,t)}{t - z - \frac{1}{4}\alpha^2}\,\mathrm{d}t, \tag{4.1}$$

and

$$\mu(z,t) := \frac{\mathrm{i}\alpha}{2^5\pi}\,\frac{(\alpha - 2\mathrm{i}(z-t)^{1/2})\,\mathrm{e}^{2\mathrm{i}(z-t)^{1/2}a}}{t^{1/2}(z-t)^{1/2}}.$$

Since the numbers 0 and $-\frac{1}{4}\alpha^2$ are branching points of the function d_a we will construct its continuation across the interval $\left(-\frac{1}{4}\alpha^2, 0\right)$ to a subset Ω_- of the lower halfplane. Let us first consider the second component ϕ_a. To find its analytical continuation to the second sheet for $\lambda \in \left(-\frac{1}{4}\alpha^2, 0\right)$ we define

$$\mu^0(\lambda,t) := \lim_{\varepsilon\to 0+} \mu(\lambda + \mathrm{i}\varepsilon, t) \qquad \text{and} \qquad I(\lambda) := \mathcal{P}\int_0^\infty \frac{\mu^0(\lambda,t)}{t-\lambda-\frac{1}{4}\alpha^2}\,\mathrm{d}t,$$

with the integral understood in the principal-value sense. We also denote

$$g_{\alpha,a}(z) := \frac{\mathrm{i}\alpha}{8}\,\frac{\mathrm{e}^{-\alpha a}}{\left(z+\frac{1}{4}\alpha^2\right)^{1/2}} \qquad \text{for} \quad z \in \Omega_- \cup \left(-\tfrac{1}{4}\alpha^2, 0\right);$$

then we are ready to formulate a lemma describing the analytic continuation of ϕ_a; we postpone its proof to appendix A.

Lemma 4.1. *The function $z \mapsto \phi_a(z)$ defined in (4.1) can be continued analytically across $\left(-\frac{1}{4}\alpha^2, 0\right)$ to a region Ω_- of the second sheet as follows:*

$$\phi_a^0(\lambda) = I(\lambda) + g_{\alpha,a}(\lambda) \qquad \text{for} \quad \lambda \in \left(-\tfrac{1}{4}\alpha^2, 0\right),$$

$$\phi_a^-(z) = -\int_0^\infty \frac{\mu(z,t)}{t-z-\frac{1}{4}\alpha^2}\,\mathrm{d}t - 2g_{\alpha,a}(z) \qquad \text{for} \quad z \in \Omega_-,\ \mathrm{Im}\,z < 0.$$

Note that apart from fixing a part of its boundary, we have imposed no restrictions on the shape of Ω_-. The lemma allows us to construct the sought analytic continuation of $d_a(\cdot)$ across the indicated segment of the real axis because the other component has no cut there. It is given by the function $\eta_a : M \mapsto \mathbb{C}$, where $M := \{z : \mathrm{Im}\, z > 0\} \cup \left(-\frac{1}{4}\alpha^2, 0\right) \cup \Omega_-$, acting as

$$\eta_a(z) = s_\beta(z) - \phi_a^{l(z)}(z),$$

where $l(z) = \pm$ if $\pm\mathrm{Im}\, z > 0$ and $l(z) = 0$ if $z \in \left(-\frac{1}{4}\alpha^2, 0\right)$, respectively; we also put $\phi_a^+ \equiv \phi_a$. The problem at hand is now to show that $\eta_a(\cdot)$ has a second-sheet zero, i.e. $\eta_a(z) = 0$ for some $z \in \Omega_-$. To proceed further it is convenient to put $\varsigma_\beta := \sqrt{-\epsilon_\beta}$, and since we are interested here primarily in large distances a, to make the following reparametrization:

$$b := \mathrm{e}^{-a\varsigma_\beta} \qquad \text{and} \qquad \tilde{\eta}(b, z) := \eta_a(z) : [0, \infty) \times M \mapsto \mathbb{C};$$

we then look for zeros of the function $\tilde{\eta}$ for small values of b. With this notation we have

$$\mu^0(\lambda, t) = \frac{\alpha}{2^5\pi} \frac{(\alpha + 2(t-\lambda)^{1/2}) b^{2(t-\lambda)^{1/2}/\varsigma_\beta}}{t^{1/2}(t-\lambda)^{1/2}}, \qquad g_{\alpha,a(b)}(\lambda) = \frac{\mathrm{i}\alpha}{8} \frac{b^{\alpha/\varsigma_\beta}}{\left(\lambda + \frac{1}{4}\alpha^2\right)^{1/2}}, \tag{4.2}$$

for $\lambda \in \left(-\frac{1}{4}\alpha^2, 0\right)$, where $a(b) := -\frac{1}{\varsigma_\beta} \ln b$, and similarly for the other constituents of $\tilde{\eta}$. This yields our main result in this section.

Theorem 4.2. *Assume that $\epsilon_\beta > -\frac{1}{4}\alpha^2$. For any b small enough the function $\tilde{\eta}(\cdot, \cdot)$ has a zero at a point $z(b) \in \Omega_-$ with the real and imaginary parts, $z(b) = \mu(b) + \mathrm{i}\nu(b)$, $\nu(b) < 0$, which in the limit $b \to 0$, i.e. $a \to \infty$, behave in the following way,*

$$\mu(b) = \epsilon_\beta + \mathcal{O}(b), \qquad \nu(b) = \mathcal{O}(b). \tag{4.3}$$

Proof. By assumption we have $\varsigma_\beta \in \left(0, \frac{1}{2}\alpha\right)$. Using formulae (4.2) together with the similar expressions of $\mu(z, t)$ and $g_{\alpha,a}(z)$ in terms of b one can check that for a fixed $b \in [0, \infty)$ the function $\tilde{\eta}(b, \cdot)$ is analytic in M, while with respect to both variables $\tilde{\eta}$ is just of the C^1 class in a neighbourhood of the point $(0, \epsilon_\beta)$. Moreover, it is easy to see that for λ close to ϵ_β the function $\phi^0_{a(b)}(\cdot)$ can be majorized by the expression Cb^M, where C, M are constants and $M > 1$. This implies $\tilde{\eta}(0, \epsilon_\beta) = 0$ and $\partial_z \tilde{\eta}(0, \epsilon_\beta) \neq 0$. Thus, by the implicit function theorem there exists a neighbourhood U_0 of zero and a unique function $z(b) : U_0 \mapsto \mathbb{C}$ such that $\tilde{\eta}(b, z(b)) = 0$ holds for all $b \in U_0$. Since $H_{\alpha,\beta}$ is self-adjoint, $\nu(b)$ cannot be positive, while $z(b) \in \left(-\frac{1}{4}\alpha^2, 0\right)$ for $b \neq 0$ can be excluded by inspecting the explicit form of $\tilde{\eta}$. Finally, by the smoothness properties of $\tilde{\eta}$ both the real and imaginary parts of $z(b)$ are of the C^1 class which yields the behaviour (4.3). □

Remark 4.3. The above theorem confirms what one expects about the behaviour of the pole using the heuristic idea about tunnelling between the point and the line, namely that the resonance width $\Gamma(b) = 2\nu(b)$ is exponentially small for a large. It is also natural to ask how the resonance pole behaves for a general a, in particular, whether it may disappear for $a \to 0$. Using the explicit formulae of lemma 4.1 one can check the following convergence:

$$|\phi_a^-(z)| \to 0 \qquad \text{as} \quad \mathrm{Im}\, z \to -\infty,$$

uniformly with respect to a. On the other hand, it is easy to see that

$$|s_\beta(z)| \to \infty \qquad \text{as} \quad \mathrm{Im}\, z \to -\infty.$$

This means that the imaginary part of $z(a)$ which represents the solution of $s_\beta(z) - \phi_a^-(z) = 0$ is a function uniformly bounded with respect to a; thus the resonance pole survives the limit $a \to 0$.

4.2. Resonances induced by broken symmetry

If there is more than one point interaction our model may exhibit another sort of resonance coming from broken symmetry. We restrict ourselves to the simplest case $n = 2$. As seen in section 3.2, the system with two point interactions placed symmetrically with respect to the line Σ and with equal coupling constants β_1, β_2 may have an embedded eigenvalue for appropriate parameter values. If we break the symmetry the corresponding resolvent pole will leave the continuous spectrum and shift to the second sheet of the analytically continued resolvent giving rise to a resonance. Of course, there are various ways how the mirror symmetry can be broken.

4.2.1. Symmetry broken by a coupling constant. Suppose first that the geometrical symmetry remains preserved, i.e. the point interactions are located at $x_1 = (0, a)$, $x_2 = (0, -a)$ with $a > 0$. The symmetry breaking will be due to unequal coupling parameters: assume that the latter are $\beta \equiv \beta_1$ and $\beta_2 = \beta + q$, where $q \in \mathbb{R}\setminus\{0\}$. To get a nontrivial result, similarly as in section 3.2 we suppose that $0 > \mu_2 > -\frac{1}{4}\alpha^2$.

To find the pole position we proceed as in section 3.2; we write the corresponding 2×2 reduced determinant, construct its analytical continuation and look for its zeros at the second sheet. This leads to the following equation:

$$\eta_q(z) = 0, \tag{4.4}$$

where

$$\eta_q(z) := s_\beta(z)(s_\beta(z) + q) - K_0(2a\sqrt{-z})^2 - (2s_\beta(z) + q)\phi_a^{l(z)}(z) - 2K_0(2a\sqrt{-z})\phi_a^{l(z)}(z)$$

and $\phi_a^{l(z)}(\cdot)$ has been defined in lemma 4.1. Our aim is to show that the function $\tilde\eta(q, z) : \mathbb{R}\setminus\{0\} \times M \to \mathbb{C}$ defined by $\tilde\eta(q, z) = \eta_q(z)$ has a zero in the lower halfplane; the set M is determined here as before, namely $M = \{z : \operatorname{Im} z > 0\} \cup \Omega_-$. Moreover, we put

$$\tilde g(\lambda) := -\mathrm{i}g_{\alpha,a}(\lambda) = \frac{\alpha}{8}\frac{\mathrm{e}^{-\alpha a}}{\left(\lambda + \frac{1}{4}\alpha^2\right)^{1/2}}$$

and use again $\kappa_2 := \sqrt{-\mu_2}$. It is also convenient to denote

$$\vartheta \equiv \vartheta(\kappa_2) := \frac{\kappa_2}{\check s_\beta'(\kappa_2) + 2aK_0'(2a\kappa_2)},$$

where the primes stand for the derivatives of the corresponding functions; with this notation we can make the following claim:

Theorem 4.4. *Suppose that $\mu_2 \in \left(-\frac{1}{4}\alpha^2, 0\right)$, then for all nonzero q small enough equation (4.4) has a solution $z(q) \in \Omega_-$ with the real and imaginary parts, $z(q) = \hat\mu(q) + \mathrm{i}\hat\nu(q)$, which are real-analytic functions of q having the following expansions:*

$$\hat\mu(q) = \mu_2 + \vartheta(\kappa_2)\, q + \mathcal{O}(q^2),$$
$$\hat\nu(q) = -\vartheta(\kappa_2)\frac{\tilde g(\mu_2)}{2\left|\check s_\beta(\kappa_2) - \phi_a^0(\mu_2)\right|^2}\, q^2 + \mathcal{O}(q^3).$$

Proof. As in theorem 4.2 we rely on the implicit function theorem, but $\tilde\eta$ is now jointly analytic, so is z_2. Since $\check s_\beta'(\kappa_2) + 2aK_0'(2a\kappa_2) > 0$ the leading term of $\hat\nu(q)$ is negative. □

Remark 4.5. The solution described in the theorem is not unique, another one comes from the symmetric eigenfunction of the corresponding Hamiltonian. This can be either a perturbed eigenvalue if μ_1 is isolated, or another resonance if μ_1 is also embedded; in the threshold case, $\mu_1 = -\frac{1}{4}\alpha^2$, the behaviour depends on the sign of q.

4.2.2. Symmetry broken by distance from the line. In contrast, assume now that the coupling strengths are the same, $\beta \equiv \beta_1 = \beta_2$, while one of the points is shifted in the perpendicular direction, $x_1 = (0, a)$ and $x_2 = (0, -a - \delta)$, where $\delta \in \mathbb{R}$. Now the equation determining the resolvent pole acquires the form

$$\begin{aligned}\breve{\eta}(\delta, z) := s_\beta(z)^2 - K_0((2a+\delta)\sqrt{-z})^2 - s_\beta(z)\big(\phi_{a+\delta}^{l(z)}(z) + \phi_a^{l(z)}(z)\big) \\ - 2K_0((2a+\delta)\sqrt{-z})\phi_{a+\delta/2}^{l(z)}(z) = 0.\end{aligned}$$

We keep the notation $\kappa_2 = \sqrt{-\mu_2}$ and also put

$$f(\delta, \kappa) = \breve{\eta}(\delta, \sqrt{-\kappa^2}).$$

Theorem 4.6. *Assume* $0 > \mu_2 > -\frac{1}{4}\alpha^2$. *For all nonzero and sufficiently small* δ *the function* $\breve{\eta}(\delta, z)$ *has a zero at a point* $z(\delta) \in \Omega_-$ *with the real and imaginary parts* $z(\delta) = \upsilon(\delta) + \mathrm{i}\iota(\delta)$ *admitting the asymptotics*

$$\upsilon(\delta) = \mu_2 - 2\kappa_2\kappa_2'\delta + \mathcal{O}(\delta^2), \qquad \iota(\delta) = -\kappa_2\kappa_2''\delta^2 + \mathcal{O}(\delta^3), \tag{4.5}$$

where

$$\kappa_2' = -\frac{2aK_0'(2a\kappa_2)}{\breve{s}_\beta'(\kappa_2) + 2aK_0'(2a\kappa_2)}, \qquad \kappa_2' = \frac{2f_{,\kappa\delta}f_{,\delta} + f_{,\kappa\kappa}\kappa_2' - f_{,\delta\delta}f_{,\kappa}}{f_{,\kappa}^2}$$

and $f_{,i}$, $f_{,ij}$ *are appropriate derivatives at the point* $\{\delta, \kappa\} = \{0, \kappa_2\}$. *Moreover, we have* $\iota(\delta) < 0$.

Proof. Similar to theorem 4.2 the argument is straightforward being based on the implicit function theorem, hence we restrict ourselves to commenting on the inequality $\iota(\delta) < 0$. Let $z(\delta) \in \big(-\frac{1}{4}\alpha^2, 0\big)$. Without losing generality we can assume $\delta > 0$ because the leading term of $\iota(\delta)$ is quadratic in δ, then $\kappa_2(\delta) = \sqrt{-z(\delta)} = \kappa_2 + \kappa_2'\delta + \mathcal{O}(\delta^2) > \kappa_2$. It is easy to see that the first and the second components of $\breve{\eta}(\delta, z(\delta))$ are real if the number $z(\delta)$ is real; furthermore, using the explicit form for $\phi_a^{l(z)}$ and properties of the exponential function one can check that

$$\operatorname{Im} f(\delta, \kappa_2(\delta)) < -2\operatorname{Im}(g_{\alpha,a+\delta/2}(z(\delta))(\breve{s}_\beta(\kappa_2(\delta)) + K_0((2a+\delta)\kappa_2(\delta))).$$

Since we have $\breve{s}_\beta(\kappa_2(\delta)) + K_0((2a+\delta)\kappa_2(\delta)) > 0$ the imaginary part of $f(\delta, \kappa_2(\delta))$ is strictly negative. Consequently, $z(\delta)$ cannot be a real number, and the possibility $\operatorname{Im} z(\delta) > 0$ is excluded by general spectral properties of self-adjoint operators. □

4.3. Scattering

While resonances in the analytically continued resolvent typically coincide with poles of the continued scattering matrix, this property does not hold automatically and has to be checked for each particular system separately. Our next goal is to illustrate it in the present setting, again in the simplest case with a single point interaction localized at the point y. To this aim we have thus to construct the S matrix for the pair $(H_{\alpha,\beta}, \tilde{H}_\alpha)$. Since the operator $H_{\alpha,\beta}$ represents a rank-one perturbation of $\tilde{H}_\alpha$, the existence and completeness of the corresponding wave operators follows immediately from the Kuroda–Birman theorem. Consequently, the S matrix is unitary; our aim is to find the on-shell S-matrix in the interval $\big(-\frac{1}{4}\alpha^2, 0\big)$, i.e. the corresponding transmission and reflection amplitudes.

4.3.1. The on-shell S matrix. Using the notation introduced above and proposition 2.3 we can write the resolvent for $\mathrm{Im}\, z > 0$ as

$$R_{\alpha,\beta}(z) = R_\alpha(z) + \eta_a(z)^{-1}(\cdot, v_z)v_z, \tag{4.6}$$

where the rank-one part in the last term is given by $v_z := R_{\alpha;L1}(z)$. We set $z = \lambda + \mathrm{i}\varepsilon$ and apply the operator $R_{\alpha,\beta}(\lambda + \mathrm{i}\varepsilon)$ to

$$\omega_{\lambda+i\varepsilon}(x) := \mathrm{e}^{\mathrm{i}(\lambda+\mathrm{i}\varepsilon+\alpha^2/4)^{1/2}x_1}\,\mathrm{e}^{-\alpha|x_2|/2},$$

then we take the limit $\varepsilon \to 0+$ in the sense of distributions. A straightforward, if tedious, calculation shows that $H_{\alpha,\beta}$ has a generalized eigenfunction which for large $|x_1|$ behaves as

$$\psi_\lambda(x) \approx \mathrm{e}^{\mathrm{i}(\lambda+\alpha^2/4)^{1/2}x_1}\,\mathrm{e}^{-\alpha|x_2|/2} + \frac{\mathrm{i}}{8}\alpha\eta_a(\lambda)^{-1}\frac{\mathrm{e}^{-\alpha a}}{\left(\lambda+\frac{1}{4}\alpha^2\right)^{1/2}}\,\mathrm{e}^{\mathrm{i}(\lambda+\alpha^2/4)^{1/2}|x_1|}\,\mathrm{e}^{-\alpha|x_2|/2} \tag{4.7}$$

for each $\lambda \in \left(-\frac{1}{4}\alpha^2, 0\right)$. To be more specific about the derivation of the above formula, one has to use again (2.3) and to rely on considerations analogous to those in the proof of lemma 4.1 to arrive at

$$v_\lambda = \lim_{\varepsilon\to 0} v_{\lambda+\mathrm{i}\varepsilon} = R_{L1}(\lambda) + S(\lambda), \tag{4.8}$$

where

$$S(\lambda) = I_\lambda(x_1, x_2) + \frac{\mathrm{i}}{8}\alpha\frac{\mathrm{e}^{-\alpha(a+|x_2|)/2}}{\left(\lambda+\frac{1}{4}\alpha^2\right)^{1/2}}\,\mathrm{e}^{\mathrm{i}(\lambda+\alpha^2/4)^{1/2}|x_1|}$$

and

$$I_\lambda(x_1, x_2) := \mathcal{P}\int_0^\infty \frac{\mu^0(\lambda, t)}{t-\lambda-\frac{1}{4}\alpha^2}\,\mathrm{e}^{-|x_2|(t-\lambda)^{1/2}}\,\mathrm{e}^{\mathrm{i}t^{1/2}x_1}\,\mathrm{d}t;$$

here $\mu^0(\lambda, t)$ is defined in section 4.1. Furthermore, note that the first component of (4.8) as well as $I_\lambda(x_1, x_2)$ vanish for $|x_1| \to \infty$, and at the same time

$$\lim_{\varepsilon\to 0}(\omega_{\lambda+\mathrm{i}\varepsilon}, v_{\lambda+\mathrm{i}\varepsilon}) = \mathrm{e}^{-\alpha a/2}.$$

In view of the results of section 4.1 and (4.6) this yields formula (4.7) which, in turn, gives the sought quantities (see also appendix B).

Proposition 4.7. *The reflection and transmission amplitudes are given by*

$$\mathcal{R}(\lambda) = \mathcal{T}(\lambda) - 1 = \frac{\mathrm{i}}{8}\alpha\eta_a(\lambda)^{-1}\frac{\mathrm{e}^{-\alpha a}}{\left(\lambda+\frac{1}{4}\alpha^2\right)^{1/2}};$$

they have the same pole in the analytical continuation to the region Ω_- as the continued resolvent.

4.4. Unstable state decay

It is also useful to look at the resonance problem from the complementary point of view and to investigate the decay of an unstable state associated with the resonance. Let us consider again the simplest case $n = 1$. The previous results tell us that if the 'unperturbed' eigenvalue ϵ_β of H_β is embedded in $\left(-\frac{1}{4}\alpha^2, 0\right)$ and a is large enough then the corresponding resonance state has a long halflife. In analogy with the Friedrichs model [4] one might expect that in the weak-coupling case, which corresponds to a large distance a here, the resonance state would be similar up to normalization to the eigenvector $\xi_0 := K_0(\sqrt{-\epsilon_\beta}\cdot)$ of H_β corresponding to ϵ_β, with the decay law being dominated by the exponential term.

However, the present model is different in one important aspect. In a typical decay problem the decaying state belongs to the absolutely continuous subspace of the Hamiltonian and thus the nondecay probability tends to zero as $t \to \infty$ by the Riemann–Lebesgue lemma [6]. Here we know from section 3.1 that $H_{\alpha,\beta}$ has always an isolated eigenvalue, and it is easy to see that the corresponding eigenfunction is *not* orthogonal to $\psi_{\alpha,\beta,a}$ for any a; it is sufficient to realize that both functions are positive, up to a possible phase factor. Consequently, the decay law $|(\xi_0, U(t)\xi_0)|^2 \|\xi_0\|^{-2}$ has always a nonzero limit as $t \to \infty$ which is equal to the squared norm of the projection of $\xi_0\|\xi_0\|^{-1}$ on the eigensubspace given by $\psi_{\alpha,\beta,a}$. On the other hand, this fact does not exclude that the decay *is* dominated by the natural exponential term as $a \to \infty$; it may happen that the nonzero limit, which certainly depends on a, is hidden in the nonexponential error term. This question requires a longer discussion and we postpone it to a subsequent publication.

5. Three dimensions: a plane and points

In analogy with the two-dimensional case investigated in the previous sections, we are now going to discuss briefly generalized Schrödinger operators in $L^2(\mathbb{R}^3)$ corresponding to the formal expression

$$-\Delta - \alpha\delta(x - \Lambda) + \sum_{i=1}^{n} \tilde{\beta}_i \delta(x - y^{(i)}), \tag{5.1}$$

where $\alpha > 0$, $\beta_i \in \mathbb{R}$ and $\Lambda := \{(\underline{x}_1, 0); \underline{x}_1 \in \mathbb{R}^2\}$ is a plane, with $y^{(i)} \in \mathbb{R}^3 \setminus \Lambda$; for the point set we will keep the same notation, $\Pi := \{y^{(i)}\}_{i=1}^n$.

5.1. Definition of Hamiltonian

To write appropriate boundary conditions let us consider functions $f \in W^{2,2}_{\mathrm{loc}}(\mathbb{R}^3 \setminus (\Lambda \cup \Pi)) \cap L^2(\mathbb{R}^3)$ which are continuous at Λ. For any such function we put $f \upharpoonright_{\mathcal{C}_{\rho,i}}$ as its restriction to the points $x \in \mathcal{C}_{\rho,i} \equiv \mathcal{C}_\rho(y_i) := \{q \in \mathbb{R}^3 : |q - y^{(i)}| = \rho\}$. In analogy with the two-dimensional case we set

$$\Xi_i(f) := \lim_{\rho\to 0} \frac{1}{\rho} f \upharpoonright_{\mathcal{C}_{\rho,i}}, \qquad \Omega_i(f) := \lim_{\rho\to 0} [f \upharpoonright_{\mathcal{C}_{\rho,i}} - \Xi_i(f)\rho]$$

for $i = 1, \ldots, n$, and

$$\Xi_\Lambda(f)(x_1) := \partial_{x_2} f(x_1, 0+) - \partial_{x_2} f(x_1, 0-), \qquad \Omega_\Lambda(f)(x_1) := f(x_1, 0),$$

and we assume that the above limits are finite and satisfy the relations

$$\Xi_i(f) = 4\pi\beta_i\Omega_i(f), \qquad \Xi_\Lambda(f)(x_1) = -\alpha\Omega_\Lambda(f)(x_1). \tag{5.2}$$

Then we define $H_{\alpha,\beta}$ as the Laplace operator with the boundary conditions given now by (5.2); it is straightforward to check that it is self-adjoint on its natural domain.

5.2. Resolvent of $H_{\alpha,\beta}$

In the three-dimensional case the free resolvent $R(z)$ with $z \in \rho(-\Delta)$ is an integral operator in $L^2(\mathbb{R}^3)$ having the kernel

$$G_z(x, x') = \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} \frac{\mathrm{e}^{\mathrm{i}p(x-x')}}{p^2 - z}\, \mathrm{d}p = \frac{\mathrm{e}^{\mathrm{i}\sqrt{z}|x-x'|}}{4\pi|x - x'|}. \tag{5.3}$$

Now we introduce the auxiliary Hilbert spaces $\mathcal{H}_0 \equiv L^2(\mathbb{R}^2)$, $\mathcal{H}_1 \equiv \mathbb{C}^n$ and abbreviate $L^2 \equiv L^2(\mathbb{R}^3)$, $W^{2,2} \equiv W^{2,2}(\mathbb{R}^3)$. By means of the trace maps $\tau_0 : W^{2,2} \to \mathcal{H}_0$ and $\tau_1 : W^{2,2} \to \mathcal{H}_1$ acting as

$$\tau_0 f := f \restriction_\Lambda, \qquad \tau_1 f := f \restriction_\Pi = \left(f \restriction_{\{y^{(1)}\}}, \ldots, f \restriction_{\{y^{(n)}\}}\right),$$

we define in analogy with (2.3) the embeddings $\mathbf{R}_{iL}(z)$, $\mathbf{R}_{Li}(z)$ and $\mathbf{R}_{ji}$. The operator-valued matrix $\Gamma(z)$ now takes the form

$$\Gamma(z) = [\Gamma_{ij}(z)] : \mathcal{H}_0 \oplus \mathcal{H}_1 \to \mathcal{H}_0 \oplus \mathcal{H}_1,$$

where $\Gamma_{ij}(z) : \mathcal{H}_i \to \mathcal{H}_j$ are the operators given by

$$\Gamma_{ij}(z)g = -\mathbf{R}_{ij}(z)g \qquad \text{for} \quad i \neq j \quad \text{and} \quad g \in \mathcal{H}_j,$$
$$\Gamma_{00}(z)f = [\alpha^{-1} - \mathbf{R}_{00}(z)]f \qquad \text{if} \quad f \in \mathcal{H}_0,$$
$$\Gamma_{11}(z)\varphi = \left[\left(\beta_l + \frac{\mathrm{i}\sqrt{z}}{4\pi}\right)\delta_{kl} - G_z(y^{(k)}, y^{(l)})(1 - \delta_{kl})\right]_{k,l=1}^{n} \varphi \qquad \text{for} \quad \varphi \in \mathcal{H}_1.$$

To describe the inverse of $\Gamma(z)$ we introduce the reduced determinant $D(z) \equiv D_{11}(z) : \mathcal{H}_1 \to \mathcal{H}_1$ given again by $D(z) = \Gamma_{11}(z) - \Gamma_{10}(z)\Gamma_{00}(z)^{-1}\Gamma_{01}(z)$ for z belonging to the resolvent set of $H_{\alpha,\beta}$. The inverse of $\Gamma(z)$ is given by $[\Gamma(z)]^{-1} : \mathcal{H}_0 \oplus \mathcal{H}_1 \to \mathcal{H}_0 \oplus \mathcal{H}_1$ defined as in (2.5). Calculations similar to those of theorem 2.2 yield the resolvent formula for $z \in \rho(H_{\alpha,\beta})$ and $\operatorname{Im} z > 0$ in the form

$$R_{\alpha,\beta}(z) \equiv (H_{\alpha,\beta} - z)^{-1} = R(z) + \sum_{i,j=0}^{1} \mathbf{R}_{Li}(z)[\Gamma(z)]_{ij}^{-1}\mathbf{R}_{jL}(z). \tag{5.4}$$

5.3. *Spectrum of $H_{\alpha,\beta}$*

Since the point interactions give rise to an explicit finite-rank perturbation to the resolvent, we find easily the absolutely continuous spectrum,

$$\sigma_{\mathrm{ess}}(H_{\alpha,\beta}) = \sigma_{\mathrm{ac}}(H_{\alpha,\beta}) = \left[-\tfrac{1}{4}\alpha^2, \infty\right).$$

As for the discrete spectrum we start again with the simplest case of a single point perturbation located at a distance a from Λ; the coupling constant of this interaction is $\beta \in \mathbb{R}$. As we have said in the introduction we will concentrate only on the differences coming from the fact that the relative dimension of the two components of the interaction support is now 2.

Let us denote by $H_\beta \equiv H_{0,\beta}$ the Laplace operator in L^2 with the perturbation supported at y only. It is well known [2] that if $\beta < 0$ then the Hamiltonian H_β has a single eigenvalue given by

$$\tilde{\epsilon}_\beta = -(4\pi\beta)^2.$$

In turn, if $\beta \geqslant 0$ the spectrum of H_β has no isolated point. However, as we will see below, the operator $H_{\alpha,\beta}$ with $\alpha > 0$ has an eigenvalue even in the latter case. To derive spectral properties of $H_{\alpha,\beta}$ we have to find solutions of the equation $\check{D}(\kappa) = 0$ for $\kappa \in \left(\frac{1}{2}\alpha, \infty\right)$, where the operator $\check{D}(\kappa)$ now acts as the multiplication by the following function:

$$\check{d}_a(\kappa) := \beta + \frac{\kappa}{4\pi} - \check{\phi}_a(\kappa)$$

with

$$\check{\phi}_a(\kappa) := \frac{\alpha}{\pi}\int_0^\infty \frac{\mathrm{e}^{-2(p^2+\kappa^2)^{1/2}a}}{(2(p^2+\kappa^2)^{1/2} - \alpha)(p^2+\kappa^2)^{1/2}}\, p\,\mathrm{d}p.$$

Since we want to investigate simultaneously the asymptotics of the eigenvalue for large and small a it is convenient to put $H_{\alpha,\beta,a} = H_{\alpha,\beta}$. We have

Theorem 5.1. *For any $\alpha > 0$ and $\beta \in \mathbb{R}$ the operator $H_{\alpha,\beta,a}$ has exactly one isolated eigenvalue $-\kappa_a^2 < -\frac{1}{4}\alpha^2$. Moreover, if $\beta > 0$ or $\tilde{\epsilon}_\beta \in \left[-\frac{1}{4}\alpha^2, \infty\right)$ then*

$$-\lim_{a\to\infty} \kappa_a^2 = \tilde{\epsilon}_\beta, \tag{5.5}$$

otherwise we have

$$-\lim_{a\to\infty} \kappa_a^2 = -\frac{1}{4}\alpha^2. \tag{5.6}$$

In distinction from the two-dimensional situation we have now

$$-\lim_{a\to 0} \kappa_a^2 = -\infty. \tag{5.7}$$

Proof. Equations (5.5) and (5.6) can be obtained by mimicking the arguments employed in the proofs of theorems 3.1 and 3.2. Using the explicit form for $\check{\phi}_a$ one can establish the existence of a positive C such that $Ca^{-1} < \check{\phi}_a(\kappa)$. It follows that $\lim_{a\to 0} \check{\phi}_a(\kappa) = \infty$ which, in turn, implies (5.7). □

Remark 5.2. In the three-dimensional case one may say that the behaviour of the eigenvalue for large a depends not only on the relation between $-\alpha^2/4$ and $\tilde{\epsilon}_\beta$; in the limit it is absorbed in the threshold also in the case when $\beta \geqslant 0$ and the discrete spectrum of H_β is empty.

Proceeding similarly as in the proof of theorem 3.5, we arrive at

Theorem 5.3. *Let $\beta = (\beta_1, \ldots, \beta_n)$, where $\beta_i \in \mathbb{R}$ and $\alpha > 0$. Operator $H_{\alpha,\beta}$ has at least one isolated eigenvalue and at most n. If all the numbers $-\beta_i$ are sufficiently large then $H_{\alpha,\beta}$ has exactly n eigenvalues.*

5.4. Resonances

To recover the resonances for the model in question we can proceed similarly as in section 4.1. Assume that $\beta < 0$ and $\tilde{\epsilon}_\beta > -\alpha^2/4$. In analogy with lemma 4.1 we state that the resolvent of $H_{\alpha,\beta}$ has a second-sheet continuation through the interval $\left(-\frac{1}{4}\alpha^2, 0\right)$. Let us put $\tilde{\varsigma}_\beta := \sqrt{-\tilde{\epsilon}_\beta} = 4\pi\beta$.

Theorem 5.4. *Assume $\tilde{\epsilon}_\beta > -\frac{1}{4}\alpha^2$. For any a sufficiently large the resolvent $R_{\alpha,\beta}$ has the second-sheet pole at a point $z(a)$ with the real and imaginary parts, $z(a) = \mu(a) + \mathrm{i}\nu(a)$, $\nu(a) < 0$, which in the limit $a \to \infty$ behave in the following way:*

$$\mu(a) = \tilde{\epsilon}_\beta + \mathcal{O}(\mathrm{e}^{-a\tilde{\varsigma}_\beta}), \qquad \nu(a) = \mathcal{O}(\mathrm{e}^{-a\tilde{\varsigma}_\beta}). \tag{5.8}$$

Remark 5.5. The resonance pole exists even if the distance is not large. In contrast to the two-dimensional case, however, the imaginary part of the pole position $\nu(a)$ diverges to $-\infty$ as $a \to 0$.

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Appendix A. Proof of lemma 4.1

In view of the edge-of-the-wedge theorem, our aim is to show that

$$\lim_{\varepsilon\to 0^+} \phi_a^{\pm}(\lambda \pm \mathrm{i}\varepsilon) = \phi_a^0(\lambda) \qquad \text{for} \quad -\tfrac{1}{4}\alpha^2 < \lambda < 0. \tag{A.1}$$

Given $\varepsilon > 0$ we put $z_\lambda^{\pm}(\varepsilon) := \lambda \pm \mathrm{i}\varepsilon$. Let $\delta(\cdot)$ be a function of the parameter ε such that $0 < \delta(\varepsilon) < \varepsilon$. We use them to define a family of the sets $C_i^{\pm}(\varepsilon)$ in the complex plane, each of which may be regarded as a graph of a curve,

$$C_1(\varepsilon) \equiv C_1^{\pm}(\varepsilon) := \{w = x : \ x \in [\delta(\varepsilon), \varepsilon^{-1}]\},$$
$$C_2^{\pm}(\varepsilon) := \{w = x \pm \mathrm{i}\varepsilon : x \in [0, x_2] \cup [x_1, \varepsilon^{-1}]\}$$

with

$$x_k \equiv x_k(\varepsilon) := \lambda + \tfrac{1}{4}\alpha^2 + (-1)^{k+1}\delta(\varepsilon), \qquad k = 1, 2;$$

furthermore,

$$C_3^{\pm}(\varepsilon) := \left\{w = z^{\pm}(\varepsilon) + \tfrac{1}{4}\alpha^2 + \delta(\varepsilon)\,\mathrm{e}^{\mathrm{i}\theta} : \theta \in \mp[0, \pi]\right\},$$
$$C_4^{\pm}(\varepsilon) := \{w = \varepsilon^{-1} \pm \mathrm{i}y : y \in [0, \varepsilon]\} \cup \{w = \pm\mathrm{i}y : y \in [\delta(\varepsilon), \varepsilon]\},$$
$$C_5^{\pm}(\varepsilon) := \left\{w = \delta(\varepsilon)\,\mathrm{e}^{\mathrm{i}\theta} : \pm\theta \in \left[0, \tfrac{1}{4}\pi\right]\right\}.$$

It is easy to see that from the definitions of $C_l^{\pm}(\varepsilon)$ that each of their unions,

$$C^{\pm}(\varepsilon) := \sum_{l=1}^{5} C_l^{\pm}(\varepsilon),$$

is a graph of a closed curve in the closed upper and lower complex halfplanes, respectively, and that the regions encircled by these loops do not contain singularities of the functions $w \mapsto \mu\big(z_\lambda^{\pm}(\varepsilon), w\big)\big(w - z_\lambda^{\pm}(\varepsilon) - \frac{1}{4}\alpha^2\big)^{-1}$; thus by the basic theorem about analytic functions we have

$$\int_{C^{\pm}(\varepsilon)} \frac{\mu\big(z_\lambda^{\pm}(\varepsilon), w\big)}{\big(w - z_\lambda^{\pm}(\varepsilon) - \frac{1}{4}\alpha^2\big)}\,\mathrm{d}w = 0. \tag{A.2}$$

This will be our starting point to check the relation (A.1):

Step 1. Since by assumption $\delta(\varepsilon) \to 0$ as $\varepsilon \to 0^+$ so $C_1(\varepsilon)$ approaches the positive real halfline, the limits we want to find are equal

$$\lim_{\varepsilon\to 0^+} \phi_a^{+}\big(z_\lambda^{+}(\varepsilon)\big) = \lim_{\varepsilon\to 0^+} \int_{C_1(\varepsilon)} \frac{\mu\big(z_\lambda^{+}(\varepsilon), w\big)}{w - z_\lambda^{+}(\varepsilon) - \frac{1}{4}\alpha^2}\,\mathrm{d}w$$

and

$$\lim_{\varepsilon\to 0^+} \phi_a^{-}(z_\lambda^{-}(\varepsilon)) = -\lim_{\varepsilon\to 0^+} \int_{C_1(\varepsilon)} \frac{\mu(z_\lambda^{-}(\varepsilon), w)}{w - z_\lambda^{-}(\varepsilon) - \frac{1}{4}\alpha^2}\,\mathrm{d}w + g_{\alpha,a}^{-}(z_\lambda^{-}(\epsilon)).$$

Step 2. Consider next the integration over $w^{\pm} = t \pm \mathrm{i}\eta(\varepsilon) \in C_2^{\pm}(\varepsilon)$. Using the following obvious convergence relations:

$$\big(z_\lambda^{\pm}(\varepsilon) - w^{\pm}\big)^{1/2} \to \mathrm{i}(t - \lambda)^{1/2} \qquad \text{as} \quad \varepsilon \to 0,$$
$$\sqrt{w^{\pm}} \to \pm\sqrt{t} \qquad \text{as} \quad \varepsilon \to 0,$$

we find

$$\lim_{\varepsilon\to 0^+}\int_{C_2^\pm(\varepsilon)} \frac{\mu\big(z_\lambda^\pm(\varepsilon), w^\pm\big)}{w^\pm - z_\lambda^\pm(\varepsilon) - \frac{\alpha^2}{4}}\,\mathrm{d}w^\pm = \mp\mathcal{P}\int_0^\infty \frac{\mu^0(\lambda, t)}{t-\lambda-\frac14\alpha^2}\,\mathrm{d}t. \tag{A.3}$$

Step 3. In the integration over the circular segments around the poles away of the origin, $w^\pm \in C_3^\pm(\varepsilon)$, we employ the convergence

$$\big(z_\lambda^\pm(\varepsilon) - w^\pm\big)^{1/2} \to \frac{\mathrm{i}}{2}\alpha \qquad \text{as} \quad \varepsilon\to 0 \qquad \sqrt{w^\pm} \to \pm\sqrt{\lambda + \frac14\alpha^2} \qquad \text{as} \quad \varepsilon \to 0,$$

which yields

$$\mu\big(z_\lambda^\pm(\varepsilon), w^\pm\big) \to \pm\frac{g_{\alpha,a}(\lambda)}{\pi\mathrm{i}} \qquad \text{as} \quad \varepsilon\to 0. \tag{A.4}$$

To proceed further we use the following identities:

$$\int_{C_3^\pm(\varepsilon)} \frac{\mu\big(z_\lambda^\pm(\varepsilon), w^\pm\big)}{w^\pm - z_\lambda^\pm(\varepsilon)-\frac14\alpha^2}\,\mathrm{d}w^\pm = \pm\frac{g_{\alpha,a}(\lambda)}{\pi\mathrm{i}}\int_{C_3^\pm(\varepsilon)} \frac{1}{w^\pm - z_\lambda^\pm(\varepsilon)-\frac14\alpha^2}\,\mathrm{d}w^\pm + \int_{C_3^\pm(\varepsilon)} \frac{\mu\big(z_\lambda^\pm(\varepsilon), w^\pm\big)\mp g_{\alpha,a}(\lambda)(\pi\mathrm{i})^{-1}}{w^\pm - z_\lambda^\pm(\varepsilon)-\frac14\alpha^2}\,\mathrm{d}w^\pm.$$

Since $\lim_{\varepsilon\to 0}\int_{C_3^\pm(\varepsilon)}\frac{1}{w^\pm - z_\lambda^\pm(\varepsilon)-\frac14\alpha^2}\,\mathrm{d}w^\pm = \mp\pi\mathrm{i}$, the limit as $\varepsilon\to 0^+$ of the first component in the above relation equals $\mp g_{\alpha,a}(\lambda)$. Moreover, in view of the convergence (A.4) and the fact that the functions involved are continuous at the segment in question we can find a function $\varepsilon\mapsto\zeta(\varepsilon)$ such that $\zeta(\varepsilon)\to 0$ as $\varepsilon\to 0$ and $\big|\mu\big(z_\lambda^\pm(\varepsilon), w^\pm\big)\mp g_{\alpha,a}(\lambda)(\pi\mathrm{i})^{-1}\big| < \zeta(\varepsilon)$ for $w^\pm\in C_3^\pm(\varepsilon)$. Then

$$\int_{C_3^\pm(\varepsilon)} \left|\frac{\mu\big(z_\lambda^\pm(\varepsilon), w^\pm\big)\mp g_{\alpha,a}(\lambda)(4\mathrm{i})^{-1}}{w^\pm - z_\lambda^\pm(\varepsilon)-\frac14\alpha^2}\right|\mathrm{d}w^\pm < \pi\zeta(\varepsilon),$$

i.e. the second integral in the above identity vanishes as $\varepsilon\to 0$. Summarizing the argument we get

$$\lim_{\varepsilon\to 0^+}\int_{C_3^\pm(\varepsilon)} \frac{\mu\big(z_\lambda^\pm(\varepsilon), w^\pm\big)}{w^\pm - z_\lambda^\pm(\varepsilon)-\frac14\alpha^2}\,\mathrm{d}w^\pm = -g_{\alpha,a}(\lambda).$$

Steps 4 and 5. Next we note that the limit $|w^\pm|\frac{\mu(z_\lambda^\pm(\varepsilon), w^\pm)}{w^\pm - z_\lambda^\pm(\varepsilon)-\frac14\alpha^2}$ as $\varepsilon\to 0$ implies for the integral over the 'vertical' parts of the integration curve

$$\lim_{\varepsilon\to 0^+}\int_{C_4^\pm(\varepsilon)} \frac{\mu\big(z_\lambda^\pm(\varepsilon), w^\pm\big)}{w^\pm - z_\lambda^\pm(\varepsilon)-\frac14\alpha^2}\,\mathrm{d}w^\pm = 0.$$

Finally, it is also easy to see that the remaining integral over $C_5^\pm(\varepsilon)$ vanishes in the limit $\varepsilon\to 0$. Combining (A.2) with the above results we get

$$\lim_{\varepsilon\to 0^+}\phi_a^\pm\big(z_\lambda^\pm(\varepsilon)\big) = \phi_a^0(\lambda),$$

so the function ϕ_a^0 is continuous for $\lambda\in\big(-\frac14\alpha^2, 0\big)$ and the proof is complete.

Appendix B. Lippmann–Schwinger equation

Here we present another possible approach to the scattering problem which we have discussed in section 4.3.

B.1. Additive representation of $H_{\alpha,\beta}$

It is also useful to write $H_{\alpha,\beta}$ in an additive form which would be reminiscent of the usual potential interaction, cf [15–17]. To this aim, let us construct for the operator $\tilde{H}_\alpha : D(\tilde{H}_\alpha) \to L^2$ the natural rigged Hilbert space, i.e. the triplet

$$\mathcal{H}_{\alpha;-} \supset L^2 \supset \mathcal{H}_{\alpha;+},$$

where $\mathcal{H}_{\alpha;\pm}$ are the completion of $D(\tilde{H}_\alpha)$ in the norm

$$\|f\|_\pm := \|(\tilde{H}_\alpha - \lambda)^{\pm 1} f\|, \qquad \text{where} \quad \lambda < -\tfrac{1}{4}\alpha^2.$$

Then we can define the extension of $\tilde{H}_\alpha$ to whole L^2; this leads to the map $\mathbf{H}_\alpha : L^2 \to \mathcal{H}_{\alpha;-}$ which expresses the canonical unitarity between L^2 and $\mathcal{H}_{\alpha;-}$. Let $D(V_\beta)$ denote the set of functions $f \in W^{2,2}_{\mathrm{loc}}(\mathbb{R}^2 \setminus (\Sigma \cup \Pi)) \cap L^2$ such that the limits $\Xi_\Sigma(f), \Omega_\Sigma(f)$ satisfy (2.1) and $\Xi_i(f), \Omega_i(f)$ are finite. Now we define the operator $V_\beta : D(V_\beta) \to \mathcal{H}_{\alpha;-}$ by

$$V_\beta \psi = \sum_{i=1}^n \psi^{\beta_i}_{\mathrm{reg}} \delta(\cdot - y^{(i)}), \qquad \text{where} \quad \psi^{\beta_i}_{\mathrm{reg}} := \begin{cases} -(2\pi\beta_i)^{-1}\Omega_i(\psi) & \text{if} \quad \beta \neq 0 \\ -\Xi_i(\psi) & \text{if} \quad \beta = 0 \end{cases}$$

Let us note that since $\mathbf{R}_{\alpha;L1} = \sum_{i=1}^n G^{(\alpha)}_z * \delta(\cdot - y^{(i)}) \in L^2$ the operator V_β is indeed well defined as a map acting to $\mathcal{H}_{\alpha;-}$. Now we can define the sought operator,

$$\tilde{H}_\alpha \hat{+} V_\beta : D(\tilde{H}_\alpha \hat{+} V_\beta) \to L^2, \qquad (\tilde{H}_\alpha \hat{+} V_\beta) f = \mathbf{H}_\alpha f + V_\beta f, \tag{B.1}$$

with the domain given by

$$D(\tilde{H}_\alpha \hat{+} V_\beta) = \{g \in D(V_\beta) : \mathbf{H}_\alpha g + V_\beta g \in L^2\}.$$

With this notation we have the following result:

Lemma B.1. $H_{\alpha,\beta} = \tilde{H}_\alpha \hat{+} V_\beta$.

Proof. It is easy to see that $\mathbf{H}_\alpha g + V_\beta g \in L^2$ if and only if $g \in D(H_{\alpha,\beta})$ because only the boundary conditions given by (2.1) ensure the appropriate compensation of $\delta(\cdot - y^{(i)})$ induced by V_β, cf [16]. At the same time, it is also easy to see that $(\tilde{H}_\alpha \hat{+} V_\beta) g(x) = \tilde{H}_\alpha g(x)$ for $x \in \mathbb{R}^2 \setminus \Pi$; this completes the proof. □

B.2. Generalized Lippman–Schwinger equation

In the same vein we now want to find an analogue of the Lippman–Schwinger equation, cf [1]. The additive representation (B.1) provides an inspiration: it is reasonable to expect that the generalized eigenvectors $\psi^\pm_\lambda$ of $H_{\alpha,\beta}$ will satisfy

$$\psi^\pm_\lambda = \omega_\lambda - R^\pm_\alpha(\lambda) V_\beta \psi^\pm_\lambda \qquad \text{for} \quad \lambda \in \left[-\tfrac{1}{4}\alpha^2, \infty\right), \tag{B.2}$$

where $\omega_\lambda = \lim_{\varepsilon\to 0} \omega_{\lambda+\mathrm{i}\varepsilon}$ are the generalized eigenvectors of H_α introduced in section 4.3.1 and $R^\pm_\alpha(\lambda)$ are the limits $\lim_{\varepsilon\to 0^+} R_\alpha(\lambda \pm \mathrm{i}\varepsilon)$ in a suitable generalized sense. We have to emphasize that equation (B.2) has only a formal meaning; our aim is now to replace it by a mathematically rigorous object. For $z^\pm(\varepsilon) = \lambda \pm \mathrm{i}\varepsilon$ define functions $\psi_{z^\pm(\varepsilon)} \in L^2$ by

$$\psi_{z^\pm(\varepsilon)} := (H_{\alpha,\beta} - z^\pm(\varepsilon))^{-1}(\tilde{H}_\alpha - z^\pm(\varepsilon))\omega_{z^+(\varepsilon)}, \tag{B.3}$$

i.e. the limits $\psi^\pm_\lambda := \lim_{\varepsilon\to 0} \psi_{z^\pm(\varepsilon)}$ in the distributional sense constitute the generalized eigenvalues of $H_{\alpha,\beta}$. Furthermore, a direct calculation shows the following relation:

$$\psi_{z^\pm(\varepsilon)} := \omega_{z^+(\varepsilon)} - R_\alpha(z^\pm(\varepsilon)) V_\beta \psi_{z^\pm(\varepsilon)}, \tag{B.4}$$

which after taking the distributional limit $\varepsilon \to 0$ gives the strict meaning to heuristic relation (B.2). Of course, the limits $\psi^\pm_\lambda$ belong only locally to L^2, however, they satisfy the same boundary conditions on $\Sigma \cup \Pi$ as functions from $D(H_{\alpha,\beta})$. This allows us to construct the extension $\bar{V}_\beta$ of V_β to $\psi_\lambda^\pm$ because the latter 'feels' only the behaviour of functions on Π. With this notation the relation (B.4) after taking the limit $\varepsilon \to 0$ acquires the following form:

$$\psi_\lambda^\pm = \omega_\lambda - R_\alpha^\pm(\lambda)\bar{V}_\beta\psi_\lambda^\pm. \tag{B.5}$$

References

[1] Albeverio S, Brasche J and Koshmanenko V 1997 Lippman–Schwinger equation for singularly perturbed operators *Methods Funct. Anal. Topol.* **3** 1–27

[2] Albeverio S, Gesztesy F, Høegh-Krohn R and Holden H 1988 *Solvable Models in Quantum Mechanics* (Heidelberg: Springer)

[3] Brasche J F and Teta A 1992 Spectral analysis and scattering theory for Schrödinger operators with an interaction supported by a regular curve *Ideas and Methods in Quantum and Statistical Physics* (Cambridge: Cambridge University Press) pp 197–211

[4] Demuth M 1976 Pole approximation and spectral concentration *Math. Nachr.* **73** 65–72

[5] Dunford N and Schwartz J 1968 *Linear Operators, II. Spectral Theory* (New York: Academic)

[6] Exner P 1985 *Open Quantum Systems and Feynman Integrals* (Dordrecht: Reidel)

[7] Exner P 2003 Spectral properties of Schrödinger operators with a strongly attractive δ interaction supported by a surface *Proc. NSF Summer Research Conference (Mt Holyoke 2002) (AMS Contemporary Mathematics Series)* (Providence, RI: American Mathematical Society) pp 25–36

[8] Exner P and Ichinose T 2001 Geometrically induced spectrum in curved leaky wires *J. Phys. A: Math. Gen.* **34** 1439–50

[9] Exner P and Kondej S 2002 Curvature-induced bound states for a δ interaction supported by a curve in $\mathbb{R}^3$ *Ann. H Poincaré* **3** 967–81

[10] Exner P and Němcová K 2003 Leaky quantum graphs: approximations by point interaction Hamiltonians *J. Phys. A: Math. Gen.* **36** 10173–93

[11] Exner P and Tater M 2004 Spectra of soft ring graphs *Waves Random Media* **14** S47–60

[12] Exner P and Yoshitomi K 2002 Asymptotics of eigenvalues of the Schrödinger operator with a strong δ-interaction on a loop *J. Geom. Phys.* **41** 344–58

[13] Exner P and Yoshitomi K 2003 Eigenvalue asymptotics for the Schrödinger operator with a δ-interaction on a punctured surface *Lett. Math. Phys.* **65** 19–26

[14] Friedrichs K O 1948 On the perturbation of continuous spectra *Commun. Pure Appl. Math.* **1** 361–406

[15] Karwowski W and Koshmanenko V 2000 The generalized Laplace operator in $L^2(\mathbb{R}^n)$ *Can. Math. Soc.* **29** 385–93

[16] Kondej S 2002 On eigenvalue problem for Schrödinger with singular perturbation *Math. Nachr.* **244** 150–69

[17] Posilicano A 2003 Self-adjoint extension by additive perturbation *Ann. Scuola Norm. Sup. Pisa Cl. Sci.* **2** 1–20

[18] Posilicano A 2004 Boundary triples and Weyl functions for singular perturbations of self-adjoint operator *Methods Funct. Anal. Topol.* at press

[19] Reed M and Simon B 1978 *Methods of Modern Mathematical Physics IV. Analysis of Operators* (New York: Academic)

[20] Weidmann J 1980 *Linear Operators in Hilbert Space* (New York: Springer)